Fibonacci sets and symmetrization in discrepancy theory

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Abstract

We study the Fibonacci sets from the point of view of their quality with respect to discrepancy and numerical integration. Let $\{b_n\}_{n=0}^{\infty}$ be the sequence of Fibonacci numbers. The b_n-point Fibonacci set $\mathcal{F}_n \subset [0,1]^2$ is defined as $\mathcal{F}_n := \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n}$, where $\{x\}$ is the fractional part of a number $x \in \mathbb{R}$. It is known that cubature formulas based on Fibonacci set $\mathcal{F}_n$ give optimal rate of error of numerical integration for certain classes of functions with mixed smoothness.

We give a Fourier analytic proof of the fact that the symmetrized Fibonacci set $\mathcal{F}'_n = \mathcal{F}_n \cup \{(p_1, 1-p_2) : (p_1,p_2) \in \mathcal{F}_n\}$ has asymptotically minimal L_2 discrepancy. This approach also yields an exact formula for this quantity, which allows us to evaluate the constant in the discrepancy estimates. Numerical computations indicate that these sets have the smallest currently known L_2 discrepancy among two-dimensional point sets.

We also introduce *quartered* L_p discrepancy which is a modification of the L_p discrepancy symmetrized with respect to the center of the unit square. We prove that the Fibonacci set $\mathcal{F}_n$ has minimal in the sense of order quartered L_p discrepancy for all $p \in (1,\infty)$. This in turn implies that certain two-fold symmetrizations of the Fibonacci set $\mathcal{F}_n$ are optimal with respect to the standard L_p discrepancy.

Keywords: Discrepancy, Fibonacci numbers, cubature formulas, numerical integration, Fourier coefficients.
2010 MSC: 11K38, 42A16, 65D30

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1. Introduction

1.1. Discrepancy

Let $\mathcal{P}_N$ be a set of N points in the unit cube $[0,1]^d$ in dimension d. The extent of uniform distribution of $\mathcal{P}_N$ can be measured by the discrepancy function:

$$D(\mathcal{P}_N, \mathbf{x}) := \#\big\{\mathcal{P}_N \cap [\mathbf{0}, \mathbf{x})\big\} - N \cdot |[\mathbf{0}, \mathbf{x})|, \tag{1.1}$$

where $\mathbf{x} = (x_1, \ldots, x_d), [\mathbf{0}, \mathbf{x}) = \prod_{j=1}^{d}[0, x_j)$, and $|\cdot|$ denotes the Lebesgue measure. The L_p norm of the above discrepancy function, usually called the L_p discrepancy, is a benchmark that one uses to evaluate the quality of a particular set of N points. The fundamental problem of the discrepancy theory is to construct sets with small L_p discrepancy.

The main principle of discrepancy theory, or theory of irregularities of distribution, states that the quantity

$$D(N, d)_p := \inf_{\mathcal{P}_N} \|D(\mathcal{P}_N, \mathbf{x})\|_p$$

must necessarily go to infinity with N when $d \geq 2$. We refer to Kuipers and Niederreiter [22], Beck and Chen [1], Matoušek [25], and Chazelle [5] for detailed surveys. The principal lower estimates for $D(N, d)_p$ are:

K. Roth's Theorem.([28], 1954) *In all dimensions $d \geq 2$, we have*

$$D(N, d)_2 \geq C(d)(\log N)^{\frac{d-1}{2}}, \tag{1.2}$$

where $C(d)$ is a positive constant that may depend on d.

W. Schmidt's Theorem. ([31], 1972) *In dimension $d = 2$,*

$$D(N, 2)_\infty \geq C \log N, \tag{1.3}$$

where C is a positive absolute constant.

Both bounds (1.2) and(1.3) are known to be sharp in the sense of order, see e.g. van der Corput [10], Davenport [11], Roth [29] and Frolov [15] for more details. One of the most famous (and relevant to our discussion) examples demonstrating sharpness of (1.3) is the irrational lattice:

$$\mathcal{A}_N(\alpha) := \left\{ \left(\frac{\mu}{N}, \{\mu\alpha\} \right) \right\}_{\mu=1}^{N}, \tag{1.4}$$

where α is an irrational number and $\{x\}$ is the fractional part of the number x. If the partial quotients of the continued fraction of α are bounded, then the L_∞ discrepancy of this set is of the order $\log N$ (see, e.g. [25], [20]). The idea of this example goes back to Lerch, 1904 [24].

In the present paper we study the distributional properties of the closely related *Fibonacci sets*. These sets are known in the theory of Quasi-Monte Carlo methods under the names *Fibonacci lattice points sets* or *Fibonacci lattice rules*, but we shall adhere to the abbreviated name. Let $\{b_n\}_{n=0}^{\infty}$ be the sequence of Fibonacci numbers:

$$b_0 = b_1 = 1, \quad b_n = b_{n-1} + b_{n-2}, \quad \text{for} \quad n \geq 2. \tag{1.5}$$

The b_n-point Fibonacci set $\mathcal{F}_n \subset [0,1]^2$ is defined as

$$\mathcal{F}_n := \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n}. \tag{1.6}$$

Obviously, for large n, the set $\mathcal{F}_n$ is close to the irrational lattice $\mathcal{A}_N(\alpha)$ with $N = b_n$ and $\alpha = \dfrac{\sqrt{5}-1}{2}$, i.e., the reciprocal of the *golden section*. It is well known (see [26]) that

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \leq C \log b_n, \tag{1.7}$$

hence, according to Schmidt's bound (1.3), Fibonacci sets also have optimal L_∞ discrepancy.

Finally, we mention another important example of a low-discrepancy construction: the van der Corput (or Hammersley) "digit-reversing" set, introduced in [10], whose L_∞ discrepancy is of the order $\log N$ (see [25] for a geometric proof). While this set is not directly related to our discussion, we shall often use it as a point of comparison.

1.2. Numerical integration

It is well known (see, for instance, [36]) that the L_∞ discrepancy (as well as other notions of discrepancy) of a finite set is closely related to the error of numerical integration with knots at the given points. We shall discuss this topic in more detail here. The quality of a set of N points for numerical integration can be measured in the following standard way. For a certain function class W compare the error of numerical integration with knots from the given set with optimal error for cubature formulas with N knots. We give a precise formulation of the problem. Numerical integration seeks good ways of approximating an integral $\int_\Omega f(\mathbf{x})d\mu$ by an expression of the form

$$\Lambda_N(f,\xi) := \sum_{j=1}^{N} \lambda_j f(\xi^j), \quad \xi = (\xi^1,\dots,\xi^N), \quad \xi^j \in \Omega, \quad j = 1,\dots,N. \tag{1.8}$$

It is clear that f has to be integrable and defined at the points $\xi^1,\dots,\xi^N$. The expression (1.8) is called a cubature formula (Λ,ξ) (in our case $\Omega \subset \mathbb{R}^2$) with knots $\xi = (\xi^1,\dots,\xi^N)$ and weights $\Lambda = (\lambda_1,\dots,\lambda_N)$. For a function class W the error of the cubature formula $\Lambda_N(\cdot,\xi)$ is defined by

$$\Lambda_N(W,\xi) := \sup_{f\in W} \left| \int_\Omega f d\mu - \Lambda_N(f,\xi) \right|. \tag{1.9}$$

In the case of equal weights $\lambda_j = 1/N$ we denote this error by $\Lambda_N^e(W,\xi)$. Set

$$\delta_N(W) := \inf_{\substack{\lambda_1,\dots,\lambda_N \\ \xi^1,\dots,\xi^N}} \Lambda_N(W,\xi); \quad \delta_N^e(W) := \inf_{\xi^1,\dots,\xi^N} \Lambda_N^e(W,\xi)$$

to be the best errors achieved by cubature formulas with N knots.

With these definitions at hand, the relation between the L_∞ discrepancy of a set $\mathcal{P}_N \subset [0,1]^2$ and the error of numerical integration with knots at $\mathcal{P}_N$ is straightforward. Define the following class of functions

$$\chi^d := \{\chi_{[\mathbf{0},\mathbf{x}]}(\mathbf{y}) := \prod_{j=1}^{d} \chi_{[0,x_j]}(y_j), \quad x_j \in [0,1], \quad j = 1,\dots,d\},$$

where $\chi_{[0,u]}(v)$ is a characteristic function of the interval $[0,u]$. Then it is clear that

$$\Lambda_N^e(\chi^d,\mathcal{P}_N) = N^{-1}\|D(\mathcal{P}_N,\mathbf{x})\|_\infty. \tag{1.10}$$

We now define classes of (periodic) functions with bounded mixed derivative, which arise naturally in numerical integration. For $r > 0$, let

$$F_r(t) := 1 + 2\sum_{k=1}^{\infty} k^{-r}\cos(2\pi kt - r\pi/2). \tag{1.11}$$

For $\mathbf{x} = (x_1, x_2)$ denote $F_r(\mathbf{x}) := F_r(x_1)F_r(x_2)$ and $MW_p^r := \{f : f = \varphi * F_r : \|\varphi\|_p \le 1\}$, where $*$ means convolution and $\|\cdot\|_p$ is the standard L_p norm.

It is known (see, for instance, survey [36]) that the Fibonacci sets $\mathcal{F}_n$ are also good for numerical integration of functions from the classes MW_p^r. The following known result gives the order of $\Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n)$ for all parameters $1 \le p \le \infty$, $r > 1/p$. In our paper, "$\asymp$" stands for "of the same order of magnitude as" and "$\ll$" stands for "less than a constant multiple of".

Theorem 1.1. *We have*

$$\Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n) \asymp \begin{cases} b_n^{-r}(\log b_n)^{1/2}, & 1 < p \le \infty, r > \max\left(\frac{1}{p}, \frac{1}{2}\right); \\ b_n^{-r}\log b_n, & p = 1, r > 1; \\ b_n^{-r}(\log b_n)^{1-r}, & 2 < p \le \infty, \frac{1}{p} < r < \frac{1}{2}; \\ b_n^{-r}((\log b_n)(\log\log b_n))^{\frac{1}{2}}, & 2 < p \le \infty, r = 1/2. \end{cases} \tag{1.12}$$

The following theorem gives the lower bounds for optimal rates of numerical integration (again, see survey [36]).

Theorem 1.2. *The following lower bound is valid for any cubature formula (Λ, ξ) with N knots $(r > 1/p)$*

$$\Lambda_N(MW_p^r, \xi) \ge C(r,p)N^{-r}(\log N)^{\frac{1}{2}}, \qquad 1 \le p < \infty.$$

The lower bounds provided by Theorem 1.2 and the upper bounds from Theorem 1.1 show that the Fibonacci cubature formulas $\Lambda_{b_n}^e(\cdot, \mathcal{F}_n)$ are optimal (in the sense of order) among all cubature formulas in the case $1 < p < \infty$, $r > \max(1/p, 1/2)$:

$$\delta_{b_n}(MW_p^r) \asymp \Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n) \asymp b_n^{-r}(\log b_n)^{1/2}.$$

We shall also make a remark in Section 2 which shows that the sets $\mathcal{F}_n$ are much better than their siblings $\mathcal{A}_N(\alpha)$ from the point of view of numerical integration of smooth functions.

It is well known (see, e.g., [36], Proposition 1.2) that the L_∞ discrepancy governs integration errors for the class MW_1^1:

$$c_1(d)\Lambda_N^e(\chi^d, \xi) \le \Lambda_N^e(MW_1^1, \xi) \le c_2(d)\Lambda_N^e(\chi^d, \xi). \tag{1.13}$$

This, together with inequality (1.7), yields the relation

$$\Lambda^e_{b_n}(MW^1_1, \mathcal{F}_n) \asymp b_n^{-1} \log b_n, \tag{1.14}$$

that was not covered by Theorem 1.1. All these results motivate us to conduct a thorough study of the Fibonacci sets.

1.3. Optimal L_2 vs. L_∞ discrepancies

At this point we would like to demonstrate that the issue of constructing sets with low L_2 discrepancy is even more subtle than in the case of L_∞. This situation is in natural contrast with the lower discrepancy estimates, where L_2 bounds are generally much simpler than L_∞.

One may be tempted to think that the optimality of the Fibonacci set $\mathcal{F}_n$ with respect to L_2 discrepancy may be implied by Theorems 1.1 and 1.2. However, this is not the case! While there is a direct relation between L_p discrepancy for $1 < p < \infty$ and the error of cubature formulas for the (non-periodic) function classes $M\dot{W}^1_{p'}(\Omega_d)$ (see [36], formula (1.15)), there is no such connection for the (periodic) classes $MW^1_{p'}$ treated in Theorem 1.1. Only the general equivalence $\delta_N(MW^1_{p'}) \asymp \delta_N(M\dot{W}^1_{p'}(\Omega_d))$ between the rates of decay of errors of optimal cubature formulas for these classes is available (see Theorem 1.1 in [36] and the remark thereafter), which is not enough to derive that $\mathcal{F}_n$ has optimal L_2 discrepancy.

Unfortunately, the L_2 discrepancy of the "classical" examples either fails to be of optimal order (the L_2 discrepancy of the N-point van der Corput set is of order $\log N$, not $\sqrt{\log N}$, [16]), or requires much more delicate arguments than L_∞ (as in the case of the Fibonacci set $\mathcal{F}_n$, [33]), or is even unknown (lattices $\mathcal{A}_N(\alpha)$ for general α).

However, discrepancy theory provides several standard ways to modify these sets in order to achieve the smallest possible order of the L_2 discrepancy and/or simplify the calculations:

1. *Cyclic shifts.* The translation idea, originated in K. Roth's papers [29], [30], was applied probabilistically to the van der Corput set. A deterministic example of such a shift was recently constructed by Bilyk [2].
2. *Digit scrambling (digit shifts).* This approach is introduced in [6] and one may refer to [25] for a comprehensive discussion and interesting constructive examples. In the past decade substantial work in this direction has been done in the context of two-dimensional low discrepancy sets, see [21], [9], [12], [3], [13].

3. *Davenport's Reflection Principle.* This idea in various guises is explored in the current paper. Roughly speaking, it states that if a finite set $\mathcal{P}_N$ has low L_∞ discrepancy, then symmetrizing this set produces a new set of low L_2 discrepancy. This approach was initiated by Davenport [11, 1956] in the case of irrational lattice. Symmetrization was subsequently used by Proinov [27], Chaix and Faure [4] for the generalized van der Corput sequences, Chen and Skriganov [8] for the van der Corput set, Larcher and Pillichshammer [23] for $(0, m, 2)$-nets and $(0, 1)$-sequences in base 2, and by other authors.

The original Davenport's construction historically was the first example demonstrating the sharpness of (1.2) (in dimension $d = 2$). His construction involved an irrational lattice $\mathcal{A}_N(\alpha)$, where α is an irrational number with bounded partial quotients, symmetrized with respect to the vertical line $x = \frac{1}{2}$. For a long time it was not clear whether this symmetrization is really necessary. The first partial answer appeared more than 20 years later. In 1979, Sós and Zaremba [33] proved that when all the partial quotients of the (finite or infinite) continued fraction of α are equal, then the set $\mathcal{A}(\alpha)$ has optimal L_2 discrepancy. In particular, this result covers the Fibonacci set $\mathcal{F}_n$ and the irrational lattice $\mathcal{A}_N((\sqrt{5}-1)/2)$ – in these cases all the partial quotients are equal to 1:

$$\|D(\mathcal{F}_n, \mathbf{x})\|_2 \asymp \|D(\mathcal{A}_{b_n}((\sqrt{5}-1)/2), \mathbf{x})\|_2 \asymp \sqrt{\log b_n}. \tag{1.15}$$

It is also suggested in the same paper that perhaps the L_2 discrepancy is not optimal for some other values of α. This means that the L_2 discrepancy depends on much finer properties of α than simply the boundedness of its partial quotients. The situation with L_p discrepancy is even less clear. These issues will be further explored in our upcoming work.

To further convince the reader of the difficulty of L_2 constructions we should mention that in higher dimensions ($d \geq 3$) explicit examples of sets with optimal order of L_2 discrepancy have been constructed only in the last few years by Chen and Skriganov [7] (simplified in [9] and extended to L_p for $p \neq 2$ by Skriganov [32]). However, the constant in the leading term of their estimate is rather large. In the two-dimensional case, Faure, Pillichshammer, Pirsic, and Schmid [13] find an effective value of this constant by considering the L_2 discrepancy of the so-called generalized Hammersley point sets.

1.4. Main results

In the present paper, we apply Davenport's symmetrization idea to the Fibonacci set. In Section 2 we prove that the symmetrized Fibonacci set $\mathcal{F}'_n$

has minimal in the sense of order L_2 discrepancy, i.e. (see Theorem 2.8)

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2 \leq C\sqrt{\log b_n}. \tag{1.16}$$

This is achieved by a meticulous examination of the Fourier coefficients of the function $D(\mathcal{F}'_n, \mathbf{x})$. This result may seem superfluous in view of the aforementioned result (1.15) of Sós and Zaremba. Nevertheless, both the result and the method present several advantages.

First of all, we are able to provide *an exact formula* allowing one to compute the precise value of L_2 norm of the discrepancy function (Theorem 2.11). This formula enabled us to computationally evaluate the constant C in the upper bound (1.16). We show that the constant we get is around 0.176006, which is better than the best previously known constant in the L_2 discrepancy upper bounds, 0.17907, provided in [13].

Unfortunately, at present we cannot compute this constant for the non-symmetrized Fibonacci set $\mathcal{F}_n$, since an analog of formulas (2.58)-(2.59) from Theorem 2.11 is not available. Technically speaking, in the non-symmetrized case certain difficulties arise in the computation of the coefficient $\widehat{D}(\mathcal{F}_n, \mathbf{0})$ (cf. Lemma 2.2) as well as $\widehat{D}(\mathcal{F}_n, \mathbf{k})$ with $\mathbf{k} = (k_1, 0)$ (cf. Lemma 2.5). This is perhaps not surprising: Davenport introduced his technique precisely to take care of the zero-order Fourier coefficient. In addition, in the case of the van der Corput set it is exactly this coefficient $\widehat{D}(\mathcal{V}_n, \mathbf{0}) = \int D(\mathcal{V}_n, \mathbf{x})\, d\mathbf{x}$ that is responsible for the large L_2 norm, see [16], [3], [2].

Finally, the proof of Sós and Zaremba was quite complicated and involved numerous ideas from number theory and probability. At the same time, our proof, which only relies on computing the Fourier coefficients of the discrepancy function, is much more transparent and opens the door to investigating more general lattices, which is the theme of our ongoing work.

In Section 3 we further develop the symmetrization idea and introduce quartered L_p discrepancy: a version of the L_p discrepancy symmetrized with respect to the center of the unit square. We prove that the Fibonacci set $\mathcal{F}_n$ has minimal in the sense of order quartered L_p discrepancy for all $p \in (1, \infty)$. While these result by itself may seem artificial, it leads to the construction of a "two-fold" symmetrization of the Fibonacci set $\mathcal{F}_n^{sym}$, which has optimal standard L_p discrepancy

$$\|D(\mathcal{F}_n^{sym}, \mathbf{x})\|_p \leq C(p)\sqrt{\log b_n} \tag{1.17}$$

for all $p \in (1, \infty)$. We note that constructions of sets with optimal L_p discrepancy for $p \neq 2$ are even more scarce than for $p = 2$. In particular, we do not know if the standard Fibonacci set $\mathcal{F}_n$ satisfies (1.17). The methods of Fourier analysis, including Littlewood-Paley theory, are applied to prove these results.

This research was supported by the National Science Foundation Grants DMS-0906260 and DMS-0801036.

2. The L_2 discrepancy of the symmetrized Fibonacci set

We shall start by briefly discussing the L_∞ discrepancy of the Fibonacci set $\mathcal{F}_n = \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n}$ and its similarities to the irrational lattice, as well as their differences, from the point of view of discrepancy and numerical integration.

As we stated in the introduction, it is a classical and over a century old result [24] that the irrational lattice

$$\mathcal{A}_N(\alpha) := \left\{ \left(\frac{\mu}{N}, \{\mu\alpha\} \right) \right\}_{\mu=1}^{N}$$

has sharp L_∞ norm if the partial quotients of the continued fraction of α are bounded. In the special case when $N = b_n$ and $\alpha = \dfrac{\sqrt{5}-1}{2}$ (the reciprocal of the *golden section*), the set $\mathcal{A}_N(\alpha)$ is closely related to the set $\mathcal{F}_n$ and satisfies the estimate

$$\|D(\mathcal{A}_n(\alpha), \mathbf{x})\|_\infty \ll \log b_n. \tag{2.18}$$

The sets $\mathcal{F}_n$ and $\mathcal{A}_N(\alpha)$ are close to each other in the following sense. For $1 \le \mu \le b_n$, the x-coordinates of the μth points of $\mathcal{F}_n$ and $\mathcal{A}_n(\alpha)$ are the same and the differences between the y-coordinates of these points are small. This follows from the well-known inequality

$$\left| \alpha - \frac{b_{n-1}}{b_n} \right| \le \frac{1}{2b_n^2}. \tag{2.19}$$

For completeness we give a simple proof of the above inequality. Consider $P(x) = x^2 + x - 1$. Then $P(\alpha) = 0$ and $|P(b_{n-1}/b_n)| = b_n^{-2}$. We have

$$|P(b_{n-1}/b_n) - P(\alpha)| = P'(\xi)|b_{n-1}/b_n - \alpha|,$$

$\xi \in \left(\frac{b_{n-1}}{b_n}, \alpha\right)$. It is easy to see that $\frac{1}{2} \le \frac{b_{n-1}}{b_n} \le \frac{2}{3}$ and $\frac{1}{2} \le \alpha \le \frac{2}{3}$. Therefore,

$$2 \le |P'(\xi)| \le \frac{7}{3}. \tag{2.20}$$

This implies (2.19). Using (2.19) we obtain

$$|\{\mu b_{n-1}/b_n\} - \{\mu\alpha\}| = |\mu b_{n-1}/b_n - \mu\alpha| \le \frac{\mu}{2b_n^2} \le \frac{1}{2b_n}. \tag{2.21}$$

(The identity above may be violated only when $\mu = b_n$, but a single point bears no significance on the results.)

As mentioned earlier, it is well known [26] that Fibbonaci sets have optimal L_∞ discrepancy:

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \ll \log b_n, \quad n \ge 2 \tag{2.22}$$

Inequality (2.21) and the following simple known lemma show that this bound can also be derived as a perturbation of (2.18).

Lemma 2.1. *Let $P_N = \{p_k\}_{k=1}^N \subset [0,1]^d$ and $Q_N = \{q_k\}_{k=1}^N \subset [0,1]^d$ be such that $\|p_k - q_k\|_\infty \le \delta$, $k = 1, \dots, N$. Then*

$$\big|\|D(P_N, \mathbf{x})\|_\infty - \|D(Q_N, \mathbf{x})\|_\infty\big| \le N\delta d.$$

The bounds (2.18) and (2.22) show that the sets $\mathcal{F}_n$ and $\mathcal{A}_n(\alpha)$ are equally good from the point of view of the L_∞ discrepancy. Theorem 1.1 from the introduction shows that the sets $\mathcal{F}_n$ are good for numerical integration. We now demonstrate by a simple example that sets $\mathcal{A}_n(\alpha)$ are not good for numerical integration of functions with high smoothness. Indeed, consider a function

$$f(x_1, x_2) := e^{2\pi i x_2}.$$

It is easy to check that $f \in MW_p^r$ for all r and $1 \le p \le \infty$. The error of numerical integration of f using $\mathcal{A}_n(\alpha)$ with equal weights $\frac{1}{b_n}$ is

$$\left|\frac{1}{b_n}\sum_{\mu=1}^{b_n} e^{2\pi i\mu\alpha}\right| = \frac{1}{b_n}\left|\frac{1 - e^{2\pi i b_n\alpha}}{1 - e^{2\pi i\alpha}}\right|.$$

Using (2.20) we get

$$\frac{3}{7} \cdot \frac{1}{b_n^2} \leq \left| \alpha - \frac{b_{n-1}}{b_n} \right| \leq \frac{1}{2b_n^2}.$$

This implies for $n \geq 3$

$$|1 - e^{2\pi i b_n \alpha}| \geq |\sin 2\pi \{b_n \alpha\}| \geq \frac{2}{\pi} \cdot 2\pi b_n \cdot \frac{3}{7} \cdot \frac{1}{b_n^2} = \frac{12}{7} \cdot \frac{1}{b_n}.$$

Therefore, the error of numerical integration of f is bounded from below by cb_n^{-2}, i.e. the error estimates do not improve when the smoothness $r > 2$. It means that the cubature formula

$$Q_{n,\alpha}(g) := \frac{1}{b_n} \sum_{q \in \mathcal{A}_n(\alpha)} g(q)$$

has a saturation property for $r > 2$. We note that this example resonates with ideas explored in [17] and [37].

We now turn our attention to the estimates for the L_2 discrepancy. Inspired by the *Davenport's Reflection Principle* [11], described in the introduction, and the similarities between the Fibonacci and irrational lattices, we symmetrize $\mathcal{F}_n$ to a $2b_n$-point set

$$\mathcal{F}_n' := \{(p_1, p_2) \cup (p_1, 1 - p_2) : (p_1, p_2) \in \mathcal{F}_n\}. \tag{2.23}$$

Its discrepancy function is

$$D(\mathcal{F}_n', \mathbf{x}) := \#\{\mathcal{F}_n' \cap [0, x_1) \times [0, x_2)\} - 2b_n x_1 x_2,$$

where $\mathbf{x} = (x_1, x_2) \in (0, 1]^2$. Rewriting it to

$$D(\mathcal{F}_n', \mathbf{x}) = \sum_{\mathbf{p}=(p_1,p_2) \in \mathcal{F}_n} \left[\chi_{[p_1,1)\times[p_2,1)}(\mathbf{x}) + \chi_{[p_1,1)\times[1-p_2,1)}(\mathbf{x})\right] - 2b_n x_1 x_2,$$

and computing the Fourier coefficients of the $D(\mathcal{F}_n', \mathbf{x})$ yields

$$\begin{aligned}\widehat{D}(\mathcal{F}_n', \mathbf{k}) = & \sum_{\mathbf{p}=(p_1,p_2) \in \mathcal{F}_n} \left[\widehat{\chi}_{[p_1,1)\times[p_2,1)}(\mathbf{k}) + \widehat{\chi}_{[p_1,1)\times[1-p_2,1)}(\mathbf{k})\right] - 2\widehat{b_n x_1 x_2} \\ = & \sum_{\mathbf{p} \in \mathcal{F}_n} \left[\int_0^1 \int_0^1 \chi_{[p_1,1)\times[p_2,1)}(x_1, x_2) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} dx_1 dx_2\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_0^1 \chi_{[p_1,1)\times[1-p_2,1)}(x_1,x_2)e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}dx_1dx_2\Big]\\
&-2b_n\int_0^1\int_0^1 x_1x_2e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}dx_1dx_2\\
= \quad & \sum_{\mathbf{p}\in\mathcal{F}_n}\Big[\int_{p_1}^1 e^{-2\pi ik_1x_1}dx_1\int_{p_2}^1 e^{-2\pi ik_2x_2}dx_2\\
&+\int_{p_1}^1 e^{-2\pi ik_1x_1}dx_1\int_{1-p_2}^1 e^{-2\pi ik_2x_2}dx_2\Big]\\
&-2b_n\int_0^1 x_1e^{-2\pi ik_1x_1}dx_1\int_0^1 x_2e^{-2\pi ik_2x_2}dx_2.
\end{aligned}
\tag{2.24}
$$

Note that

$$
\sum_{\mu=1}^{b_n} e^{-2\pi il\mu/b_n} = \begin{cases} b_n, & l\equiv 0 \pmod{b_n},\\ 0, & l\not\equiv 0 \pmod{b_n}.\end{cases} \tag{2.25}
$$

Let $L(n) := \{\mathbf{k}=(k_1,k_2)\in\mathbf{Z}^2 : k_1+b_{n-1}k_2\equiv 0 \pmod{b_n}\}$, then

$$
\sum_{\mu=1}^{b_n} e^{-2\pi i(k_1+b_{n-1}k_2)\mu/b_n} = \begin{cases} b_n, & (k_1,k_2)\in L(n),\\ 0, & (k_1,k_2)\notin L(n).\end{cases} \tag{2.26}
$$

Now let us consider different cases:

Case 1. $k_1=0$, $k_2=0$. We have the following lemma:

Lemma 2.2. $\widehat{D}(\mathcal{F}_n',\mathbf{0}) = -\dfrac{1}{2}$.

Proof. From (2.24) we get

$$
\begin{aligned}
\widehat{D}(\mathcal{F}_n',\mathbf{0}) &= \sum_{\mathbf{p}\in\mathcal{F}_\mathbf{n}}\big[(1-p_1)(1-p_2)+(1-p_1)p_2\big]-\frac{b_n}{2}\\
&= \sum_{\mathbf{p}\in\mathcal{F}_\mathbf{n}}\big(1-p_1\big)-\frac{b_n}{2}\\
&= \sum_{\mu=1}^{b_n}(1-\mu/b_n)-\frac{b_n}{2}
\end{aligned}
$$

$$\begin{aligned} &= b_n - \frac{b_n(b_n+1)}{2b_n} - \frac{b_n}{2} \\ &= -\frac{1}{2}. \end{aligned} \tag{2.27}$$

Case 2. $k_1 \neq 0$, $k_2 \neq 0$.
In this case

$$\begin{aligned} \widehat{D}(\mathcal{F}_n', \mathbf{k}) &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \big[(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2 p_2}) \\ &\quad +(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2(1-p_2)})\big] + \frac{b_n}{2\pi^2 k_1 k_2} \\ &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \big[(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2 p_2}) \\ &\quad +(1-e^{-2\pi i k_1 p_1})(1-e^{2\pi i k_2 p_2})\big] + \frac{b_n}{2\pi^2 k_1 k_2}. \end{aligned} \tag{2.28}$$

Then we have the following lemma:

Lemma 2.3. *If $k_1 \neq 0$, $k_2 \neq 0$, then*

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \frac{b_n}{2\pi^2 k_1 k_2} \tag{2.29}$$

provided that at least one of k_1 and k_2 is 0 modulo b_n.

Proof. Without loss of generality assume $k_1 \equiv 0 \pmod{b_n}$, then

$$e^{-2\pi i k_1 p_1} = e^{\frac{-2\pi i k_1 \mu}{b_n}} = 1.$$

So from (2.28) we get

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \frac{b_n}{2\pi^2 k_1 k_2}. \tag{2.30}$$

Lemma 2.4. *Assume $k_1 \not\equiv 0 \pmod{b_n}$ and $k_2 \not\equiv 0 \pmod{b_n}$, then*

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \begin{cases} \frac{-b_n}{2\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \equiv 0,\ k_1 - k_2 b_{n-1} \equiv 0, \\ \frac{-b_n}{4\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \equiv 0,\ k_1 - k_2 b_{n-1} \not\equiv 0, \\ \frac{-b_n}{4\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \not\equiv 0,\ k_1 - k_2 b_{n-1} \equiv 0, \\ 0, & k_1 + k_2 b_{n-1} \not\equiv 0,\ k_1 - k_2 b_{n-1} \not\equiv 0, \end{cases} \tag{2.31}$$

where all congruences are taken modulo b_n.

Proof. Since by (2.25) $\sum_{\mathbf{p}\in\mathcal{F}_n} e^{\pm 2\pi x i k_j p_j} = 0$ for $j = 1, 2$, we can rewrite (2.28) as

$$\begin{aligned}
\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[2 + e^{-2\pi i(k_1p_1+k_2p_2)} + e^{-2\pi i(k_1p_1-k_2p_2)}\right] + \frac{b_n}{2\pi^2 k_1 k_2} \\
&= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[e^{-2\pi i(k_1p_1+k_2p_2)} + e^{-2\pi i(k_1p_1-k_2p_2)}\right] \\
&= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mu=1}^{b_n} \left[e^{\frac{-2\pi i\mu(k_1+k_2b_{n-1})}{b_n}} + e^{\frac{-2\pi i\mu(k_1-k_2b_{n-1})}{b_n}}\right]. \qquad (2.32)
\end{aligned}$$

If both $k_1 + k_2 b_{n-1} \equiv 0 \pmod{b_n}$ and $k_1 - k_2 b_{n-1} \equiv 0 \pmod{b_n}$ hold, i.e. $(k_1, k_2) \in L(n)$ and $(k_1, -k_2) \in L(n)$, we get

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \frac{-b_n}{2\pi^2 k_1 k_2}. \qquad (2.33)$$

Note that for odd b_n the congruences $k_1+k_2b_{n-1} \equiv 0 \pmod{b_n}$, $k_1-k_2b_{n-1} \equiv 0 \pmod{b_n}$ imply $k_1 \equiv 0 \pmod{b_n}$ that violates the assumptions of Lemma 3.3. Thus this case is possible only for even b_n.

If only one of $k_1 + k_2b_{n-1} \equiv 0 \pmod{b_n}$, $k_1 - k_2b_{n-1} \equiv 0 \pmod{b_n}$ holds, or in other words only one of (k_1, k_2), $(k_1, -k_2)$ is in $L(n)$, then

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \frac{-b_n}{4\pi^2 k_1 k_2}. \qquad (2.34)$$

If $k_1 + k_2b_{n-1} \not\equiv 0 \pmod{b_n}$ and $k_1 - k_2b_{n-1} \not\equiv 0 \pmod{b_n}$, i.e. both (k_1, k_2) and $(k_1, -k_2)$ are not in $L(n)$, then we get

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = 0. \qquad (2.35)$$

Case 3. $k_1 \neq 0$, $k_2 = 0$. We have the following lemma:

Lemma 2.5. *If $k_1 \neq 0, k_2 = 0$,*

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \begin{cases} \dfrac{b_n}{2\pi i k_1}, & k_1 \equiv 0 \pmod{b_n}, \\ 0, & k_1 \not\equiv 0 \pmod{b_n}. \end{cases} \qquad (2.36)$$

Proof. We obtain from (2.24),

$$\begin{aligned}\widehat{D}(\mathcal{F}_n', \mathbf{k}) &= \frac{-1}{2\pi i k_1} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[(1-e^{-2\pi i k_1 p_1})(1-p_2) + (1-e^{-2\pi i k_1 p_1})p_2\right] + \frac{b_n}{2\pi i k_1} \\ &= \frac{-1}{2\pi i k_1} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[1-e^{-2\pi i k_1 p_1}\right] + \frac{b_n}{2\pi i k_1}. \qquad (2.37)\end{aligned}$$

If $k_1 \equiv 0 \pmod{b_n}$, then $e^{-2\pi i k_1 p_1} = 1$, thus

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \frac{b_n}{2\pi i k_1}. \qquad (2.38)$$

If $k_1 \not\equiv 0 \pmod{b_n}$, then $\sum_{\mathbf{p}\in\mathcal{F}_n} e^{-2\pi i k_1 p_1} = 0$, hence

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = 0. \qquad (2.39)$$

Case 4. $k_1 = 0, k_2 \neq 0$. We have the following lemma:

Lemma 2.6. *If* $k_1 = 0, k_2 \neq 0$,

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \begin{cases} \dfrac{b_n}{2\pi i k_2}, & k_2 \equiv 0 \pmod{b_n}, \\ 0, & k_2 \not\equiv 0 \pmod{b_n}. \end{cases} \qquad (2.40)$$

Proof. From (2.24) we obtain

$$\begin{aligned}\widehat{D}(\mathcal{F}_n', \mathbf{k}) &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[(1-p_1)(1-e^{-2\pi i k_2 p_2}) + (1-p_1)(1-e^{2\pi i k_2 p_2})\right] \\ &\quad + \frac{b_n}{2\pi i k_2} \\ &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[(1-p_1)(2-e^{-2\pi i k_2 p_2} - e^{2\pi i k_2 p_2})\right] + \frac{b_n}{2\pi i k_2}.\end{aligned}$$

If $k_2 \equiv 0 \pmod{b_n}$, then $e^{\pm 2\pi i k_2 p_2} = 1$, and

$$\widehat{D}(\mathcal{F}_n', \mathbf{k}) = \frac{b_n}{2\pi i k_2}. \qquad (2.41)$$

If $k_2 \not\equiv 0 \pmod{b_n}$, then $\sum_{\mathbf{p}\in\mathcal{F}_n} e^{\pm 2\pi i k_2 p_2} = 0$, and we get

$$\begin{aligned}
\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[2 - 2p_1 + p_1 e^{-2\pi i k_2 p_2} + p_1 e^{2\pi i k_2 p_2}\right] + \frac{b_n}{2\pi i k_2} \\
&= \frac{-1}{2\pi i k_2} \sum_{\mu=1}^{b_n} \left[2 - 2\frac{\mu}{b_n} + \frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right] + \frac{b_n}{2\pi i k_2} \\
&= \frac{-1}{2\pi i k_2} \left[2b_n - 2b_n - 1 + \sum_{\mu=1}^{b_n} \left(\frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right)\right] \\
&= \frac{1}{2\pi i k_2} + \frac{-1}{2\pi i k_2} \left[\sum_{\mu=0}^{b_n-1} \left(\frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right) + 2\right].
\end{aligned} \tag{2.42}$$

Let us set

$$f(x) = \sum_{\mu=0}^{b_n-1} e^{\frac{2\pi i \mu x}{b_n}} = \frac{e^{2\pi i x} - 1}{e^{\frac{2\pi i x}{b_n}} - 1}.$$

On one hand,

$$f'(x) = \sum_{\mu=0}^{b_n-1} \frac{2\pi i \mu}{b_n} e^{\frac{2\pi i \mu x}{b_n}}, \tag{2.43}$$

and thus

$$f'(k_2 b_{n-1}) = \sum_{\mu=0}^{b_n-1} \frac{2\pi i \mu}{b_n} e^{\frac{2\pi i \mu k_2 b_{n-1}}{b_n}}; \tag{2.44}$$

on the other hand

$$f'(x) = \frac{2\pi i e^{2\pi i x}(e^{\frac{2\pi i x}{b_n}} - 1) - (e^{2\pi i x} - 1)\frac{2\pi i}{b_n} e^{\frac{2\pi i x}{b_n}}}{\left(e^{\frac{2\pi i x}{b_n}} - 1\right)^2}. \tag{2.45}$$

Note that $e^{2\pi i k_2 b_{n-1}} = 1$ and thus

$$\begin{aligned}
f'(k_2 b_{n-1}) &= \frac{2\pi i (e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1)}{\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right)^2} \\
&= \frac{2\pi i}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1}.
\end{aligned} \tag{2.46}$$

Comparing (2.44) and (2.46) we find

$$\sum_{\mu=0}^{b_{n-1}} \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} = \frac{1}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1}.$$

In the same way we get

$$\sum_{\mu=0}^{b_{n-1}} \frac{\mu}{b_n} e^{\frac{-2\pi i k_2 \mu b_{n-1}}{b_n}} = \frac{1}{e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1}.$$

Therefore,

$$\begin{aligned}
\sum_{\mu=0}^{b_{n-1}} \left[\frac{\mu}{b_n} e^{\frac{-2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} \right] &= \frac{1}{e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1} + \frac{1}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1} \\
&= \frac{\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right) + \left(e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1\right)}{\left(e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1\right)\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right)} \\
&= \frac{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} + e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 2}{2 - e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - e^{\frac{2\pi i k_2 b_{n-1}}{b_n}}} \\
&= -1.
\end{aligned}$$

Hence from (2.42)

$$\begin{aligned}
\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{1}{2\pi i k_2} + \frac{-1}{2\pi i k_2}(-1+2) \\
&= 0. \qquad (2.47)
\end{aligned}$$

Remark 2.7. *We define the sets*

$$\begin{aligned}
S_1 &= \{(k_1, k_2) : k_1, k_2 \neq 0, \ k_1 \equiv 0 \pmod{b_n}\}, \\
S_2 &= \{(k_1, k_2) : k_1, k_2 \neq 0, \ k_2 \equiv 0 \pmod{b_n}\}, \\
S_3 &= \{(k_1, 0) : k_1 \equiv 0 \pmod{b_n}, \quad k_1 \neq 0\}, \\
S_4 &= \{(0, k_2) : k_2 \equiv 0 \pmod{b_n}, \quad k_2 \neq 0\}, \\
S_5 &= \{(k_1, k_2) : (k_1, k_2) \in L(n) \setminus \{\mathbf{0}\}, \quad k_1, k_2 \not\equiv 0 \pmod{b_n}\}, \\
S_6 &= \{(k_1, k_2) : (k_1, -k_2) \in L(n) \setminus \{\mathbf{0}\}, \quad k_1, k_2 \not\equiv 0 \pmod{b_n}\}.
\end{aligned}$$

Based on previous lemmas, we have the following observations. The results of lemmas 2.3, 2.4, 2.5, and 2.6 imply that for $\mathbf{k} \in S_1 \cup ... \cup S_6$ *we have*

$$|\widehat{D}(\mathcal{F}'_n, \mathbf{k})| \ll \frac{b_n}{\prod_{j=1}^{2} \max(|k_j|, 1)}. \tag{2.48}$$

In all other cases, the corresponding Fourier coefficients are equal to zero, see (2.35), (2.39) *and* (2.47).

For $\mathbf{k} \in S_1$, we write $k_1 = lb_n$, where $l \in \mathbb{Z} \setminus \{0\}$. Then $|\widehat{D}(\mathcal{F}'_n, \mathbf{k})| = \frac{1}{2\pi^2 |k_1 l|}$. We deal with S_2, S_3, and S_4 similarly. We are now ready to proceed to the main theorem.

Theorem 2.8. *For the symmetrized Fibonacci set* $\mathcal{F}'_n \subset [0, 1]^2$, *we have*

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2 \ll \sqrt{\log b_n}. \tag{2.49}$$

Proof. By Parseval's theorem,

$$\begin{aligned} \|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 = \|\widehat{D}(\mathcal{F}'_n, \mathbf{k})\|_2^2 \;&\leq\; |\widehat{D}(\mathcal{F}'_n, \mathbf{0})|^2 + \sum_{i=1}^{6} \sum_{\mathbf{k} \in S_i} |\widehat{D}(\mathcal{F}'_n, \mathbf{k})|^2 \\ &\ll\; \sum_{\mathbf{k} \in L(n) \setminus \{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2} \max(k_j^2, 1)} \\ &\quad + \sum_{(k_1, -k_2) \in L(n) \setminus \{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2} \max(k_j^2, 1)} \\ &\quad + 2 \sum_{l \neq 0} \sum_{k \neq 0} \frac{1}{(kl)^2} + 2 \sum_{l \neq 0} \frac{1}{l^2}. \end{aligned}$$

It is easy to see that the last two sums converge to some constants and the first two are completely similar to each other. We can thus estimate

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 \ll \sum_{\mathbf{k}\in L(n)\setminus\{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2} \max(k_j^2, 1)}. \tag{2.50}$$

We now use the following lemma, see Lemma 2.1 from Chapter 4 of [35].

Lemma 2.9. *Denote*

$$\Gamma(N) := \left\{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbf{Z}^d : \prod_{j=1}^{d} \max(|k_j|, 1) \leq N\right\}$$

and

$$Z_l := \left(\Gamma(2^{l+1}\gamma b_n) \setminus \Gamma(2^l \gamma b_n)\right) \cap L(n), \qquad l = 0, 1, 2, \ldots,$$

then there exists an absolute constant $\gamma > 0$ *such that for any* $n > 2$

$$\Gamma(\gamma b_n) \cap \left(L(n) \setminus \mathbf{0}\right) = \emptyset,$$

and

$$|Z_l| \ll 2^l (l+1) \log b_n, \qquad l = 0, 1, 2, \ldots. \tag{2.51}$$

Therefore, the summation in (2.50) can be estimated as

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 \ll \sum_{l\geq 0} \sum_{\mathbf{k}\in Z_l} \frac{1}{|2^l|^2}, \tag{2.52}$$

and using the cardinality estimate of Z_l in (2.51), we get,

$$\begin{aligned} \|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 &\ll \sum_{l\geq 0} \frac{2^l (l+1) \log b_n}{(2^l)^2} \\ &= \log b_n \sum_{l\geq 0} \frac{l+1}{2^l} \\ &\ll \log b_n. \end{aligned}$$

Hence

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2 \ll \sqrt{\log b_n}.$$

Remark 2.10. *In this section we symmetrize the original Fibonacci set to obtain a* $2b_n-$*point set* $\mathcal{F}_n' = \{(p_1,p_2) \cup \{(p_1, 1-p_2) : (p_1,p_2) \in \mathcal{F}_n\}$*. Obviously, the* L_∞ *discrepancy of* $\mathcal{F}_n'$ *satisfies the same upper bound as* $\mathcal{F}_n$ *in the order of magnitude and thus is optimal. Theorem 2.8 verifies the sharpness of its* L_2 *discrepancy.*

In fact, we can also demonstrate that a $4b_n-$*point set* $\mathcal{F}_n'' = \{(p_1,p_2) \cup (1-p_1,p_2) \cup \{(p_1,1-p_2) \cup \{(1-p_1,1-p_2) : (p_1,p_2) \in \mathcal{F}_n\}$ *achieves the minimal* L_2 *discrepancy as well. The computation is completely analogous, and, in Case 4 (Lemma 2.6), it is much more straightforward.*

Next, we derive a formula which provides the exact value of $\|D(\mathcal{F}_n', \mathbf{x})\|_2$. For simplicity, we shall first assume that b_n is odd, and thus $S_5 \cap S_6 = \emptyset$. We start with the contribution of $\mathbf{k} \in S_5$, using the notation introduced in Remark 2.7. In this case, $\widehat{D}(\mathcal{F}'{}_n, \mathbf{k}) = -\frac{b_n}{4\pi^2 k_1 k_2}$. We shall make use of the well-known identity (see e.g. [34], page 165, ex. 15):

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+x)^2} = \frac{\pi^2}{\sin^2(\pi x)}. \tag{2.53}$$

Denote $k_1 + k_2 b_{n-1} = l b_n$, for $l \in \mathbb{Z}$ and toward the end of the computation write $k_2 = m b_n + r$, where $m \in \mathbb{Z}$ and $r = 1, ..., b_n - 1$. We have, by Lemma 2.4

$$\begin{aligned}
\sum_{k\in S_5} \left|\widehat{D}(\mathcal{F}'{}_n, \mathbf{k})\right|^2 &= \frac{b_n^2}{16\pi^4} \sum_{k_2 \not\equiv 0 \bmod b_n} \frac{1}{k_2^2} \sum_{l\in\mathbb{Z}} \frac{1}{b_n^2} \cdot \frac{1}{\left(l - \frac{b_{n-1}k_2}{b_n}\right)^2} \\
&= \frac{1}{16\pi^2} \sum_{k_2 \not\equiv 0 \bmod b_n} \frac{1}{k_2^2 \sin^2\left(\frac{\pi b_{n-1} k_2}{b_n}\right)} \\
&= \frac{1}{16\pi^2} \sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right)} \sum_{m\in\mathbb{Z}} \frac{1}{b_n^2} \cdot \frac{1}{\left(m + \frac{r}{b_n}\right)^2} \\
&= \frac{1}{16 b_n^2} \sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right) \cdot \sin^2\left(\frac{\pi r}{b_n}\right)},
\end{aligned} \tag{2.54}$$

where we have used identity (2.53) in the second and the last equalities above. It is obvious that the contribution of $\mathbf{k} \in S_6$ is identical. If b_n is even, a "correction term" $\frac{1}{8b_n^2}$ arises due to the fact that $S_5 \cap S_6 \neq \emptyset$ (we leave the computation to the reader).

Using the inclusion-exclusion principle and the identity

$$\sum_{l\in\mathbb{N}} \frac{1}{l^2} = \frac{\pi^2}{6}, \tag{2.55}$$

we obtain by Lemma 2.3

$$\begin{aligned}\sum_{k\in S_1\cup S_2} \left|\widehat{D}(\mathcal{F}'_n,\mathbf{k})\right|^2 &= 4\sum_{l_1\in\mathbb{N},k_2\in\mathbb{N}} \frac{b_n^2}{4\pi^4\cdot l_1^2 b_n^2\cdot k_2^2} + 4\sum_{k_1\in\mathbb{N},l_2\in\mathbb{N}} \frac{b_n^2}{4\pi^4\cdot k_1^2\cdot l_2^2 b_n^2} \\ &\quad -4\sum_{l_1\in\mathbb{N},l_2\in\mathbb{N}} \frac{b_n^2}{4\pi^4 b_n^4 l_1^2 l_2^2} \\ &= 8\cdot\frac{1}{4\pi^4}\cdot\frac{\pi^2}{6}\cdot\frac{\pi^2}{6} - 4\frac{1}{144b_n^2} = \frac{1}{36}\left(2-\frac{1}{b_n^2}\right). \end{aligned} \tag{2.56}$$

(The multiplication by 4 above accounts for all possible choices of signs).

Finally, Lemmas 2.5 and 2.6 yield

$$\sum_{k\in S_3\cup S_4} \left|\widehat{D}(\mathcal{F}'_n,\mathbf{k})\right|^2 = 2\cdot\frac{b_n^2}{4\pi^2}\sum_{l\in\mathbb{Z}\setminus\{0\}} \frac{1}{b_n^2 l^2} = \frac{1}{6}. \tag{2.57}$$

Putting together equations (2.54), (2.56), and (2.57), and the relation $\widehat{D}(\mathcal{F}'_n,\mathbf{0}) = -\frac{1}{2}$ (Lemma 2.2) we obtain

Theorem 2.11. *For $n \geq 2$ we have*

$$\|D(\mathcal{F}'_n,\mathbf{x})\|_2^2 = \frac{1}{8b_n^2}\sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right)\cdot\sin^2\left(\frac{\pi r}{b_n}\right)} + \frac{17}{36} - \frac{1}{36b_n^2} \quad \text{when } b_n \text{ is odd }, \tag{2.58}$$

$$\|D(\mathcal{F}'_n,\mathbf{x})\|_2^2 = \frac{1}{8b_n^2}\sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right)\cdot\sin^2\left(\frac{\pi r}{b_n}\right)} + \frac{17}{36} + \frac{7}{72b_n^2} \quad \text{when } b_n \text{ is even.} \tag{2.59}$$

We should recall that the L^2 discrepancy of an arbitrary N-point set can be computed precisely using Warnock's formula [38]. However, the fastest known way to perform this computation requires $\mathcal{O}(N\log N)$ steps [18], [14]

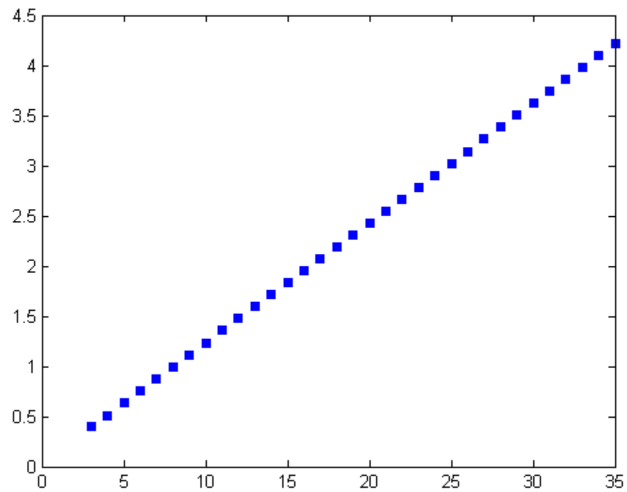

Figure 1: The values of S_n for $n \leq 35$.

(see also the discussion in §2.4 of [25]). The formulas of Theorem 2.11 require only of the order of $b_n \asymp N$ steps to compute the discrepancy of the symmetrized Fibonacci set $\mathcal{F}'_n$.

It can be shown directly that the main term in equations (2.58) and (2.59) is of the order $\log b_n \asymp n$. Besides, numerical experiments indicate that

$$S_n = \frac{1}{b_n^2} \sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right) \cdot \sin^2\left(\frac{\pi r}{b_n}\right)} \approx 0.119257 \cdot n. \tag{2.60}$$

We have performed these computations using MATLAB and Maple up to $n = 35$, which corresponds to $N = 2b_{35} = 29,860,704$. The differences between successive values of S_n stabilize very quickly (up to the sixth decimal digit starting with $n = 16$, see Table 1 and Figure 1). Straightforward computations become unstable and too slow beyond this value; in particular, very time-consuming computations for $36 \leq n \leq 40$ yielded consecutive differences between 0.119240 and 0.119265. We plan to conduct more sophisticated and precise calculations in the future. We are extremely grateful and indebted to Douglas Meade for his help with the numerical experiments.

Since the symmetrized Fibonacci set $\mathcal{F}'_n$ has $N = 2b_n$ points, we have

$$\lim_{n\to\infty} \frac{\log N}{n} = \log\left(\frac{\sqrt{5}+1}{2}\right) \approx 0.481212.$$

Here "log" stands for the natural logarithm in order to compare our results with the upper bound in [13]. Assuming that the results of the numerical experiments are indeed true, we obtain

$$\begin{aligned} \|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 &= \frac{1}{8b_n^2} \sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right) \cdot \sin^2\left(\frac{\pi r}{b_n}\right)} + \mathcal{O}(1) \qquad (2.61) \\ &= (0.125) \cdot (0.119257...) \cdot n + \mathcal{O}(1) \\ &= 0.030978... \cdot \log N + o(\log N). \end{aligned}$$

This (numerically obtained) constant 0.030978 above is smaller than the analogous best constant, 0.03206, found in [13] for the scrambled generalized Hammersley point sets. Hence, numerical computations indicate that among all two-dimensional point sets, the symmetrized Fibonacci lattice has the smallest known L_2 discrepancy:

Corollary 2.12. *The symmetrized Fibonacci sets $\mathcal{F}'_n$ with $N = 2b_n$ points satisfy:*

$$\lim_{n\to\infty} \frac{\|D(\mathcal{F}'_n, \mathbf{x})\|_2}{\sqrt{\log N}} \approx \sqrt{0.030978} \approx 0.176006. \qquad (2.62)$$

The previously best known constant, obtained in [13] is slightly larger, 0.17907. However, our corollary, strictly speaking, is not a mathematical fact, but rather a result of experiments. The actual values of the L^2 discrepancy provided by (2.58) and (2.59) for moderate values of n are somewhat larger. For example, for $n = 35$, i.e. $N = 29,860,704$, we have $\frac{\|D(\mathcal{F}'_n,\mathbf{x})\|_2}{\sqrt{\log N}} \approx 0.240969$ (see Table 1 for a full list of values).

It is worth mentioning that the best currently known constant in the lower estimates was found by Hinrichs and Markhasin [19]. They prove that, in our notation, $D(N,2)_2 \geq \sqrt{\frac{1}{2^{16}\log 2}}\sqrt{\log N} \approx 0.0046918 \cdot \sqrt{\log N}$.

Table 1: The results of numerical computations

n	$N = 2b_n$	S_n	$\|D(\mathcal{F}'_n, \mathbf{x})\|_2^2$	$\frac{\|D(\mathcal{F}'_n, \mathbf{x})\|_2}{\sqrt{\log N}}$
15	1974	1.832556	0.701292	0.304012
16	3194	1.951812	0.716199	0.297924
17	5168	2.071070	0.731106	0.292416
18	8362	2.190327	0.746013	0.287405
19	13530	2.309584	0.760920	0.282825
20	21892	2.428840	0.775827	0.278622
21	35422	2.548097	0.790734	0.274749
22	57314	2.667354	0.805642	0.271168
23	92736	2.786611	0.820549	0.267847
24	150050	2.905868	0.835456	0.264757
25	242786	3.025125	0.850363	0.261874
26	392836	3.144382	0.865270	0.259178
27	635622	3.263639	0.880177	0.256651
28	1028458	3.382896	0.895084	0.254277
29	1664080	3.502153	0.909991	0.252043
30	2692538	3.621410	0.924898	0.249936
31	4356618	3.740667	0.939806	0.247945
32	7049156	3.859924	0.954713	0.246061
33	11405774	3.979181	0.969620	0.244275
34	18454930	4.098438	0.984527	0.242580
35	29860704	4.217695	0.999434	0.240969

3. Quartered L_p discrepancy and two-fold symmetrization

We shall consider a modification of the classical L_p discrepancy function. For a parameter $a \in [0, 1/2]$ define the following univariate characteristic function for $t \in [0, 1)$.

$$S(a, t) := \chi_{[1/2-a,1/2+a]}(t),$$

and for the multivariate case $\mathbf{x} \in [0, 1/2]^d$, $\mathbf{y} \in [0, 1]^d$

$$S(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^{d} S(x_j, y_j).$$

For a set $\xi := \{\xi^\mu\}_{\mu=1}^N \subset [0,1]^d$, define *the quartered L_p discrepancy* as follows

$$D^q(\xi, N, d)_p := \left\| \sum_{\mu=1}^N S(\mathbf{x}, \xi^\mu) - N \int_{[0,1]^d} S(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right\|_{L_p([0,1/2]^d, \mathbf{x})}. \tag{3.63}$$

The expression inside the norm is simply the discrepancy of ξ with respect to the box centered at $\mathbf{1/2} = (1/2, ..., 1/2)$ and opposite corners at $\mathbf{1/2} \pm \mathbf{x}$. Let us note that this notion of discrepancy does not quite measure the uniformity of distribution of ξ as it doesn't change when we move all points to the same quadrant with respect to the center of the square. However, precisely these considerations relate the quartered L_p discrepancy and standard L_p discrepancy. We have

$$S(a,t) = \chi_{[0,\frac{1}{2}+a]}(t) - \chi_{[0,\frac{1}{2}-a]}(t).$$

This allows us to obtain the following inequality

$$D^q(\xi, N, d)_p \leq 2^d \|D(\xi, \mathbf{x})\|_p.$$

The quartered L_p discrepancy can be bounded from below by the L_p discrepancy of a symmetrized set ξ^{sym}, that we define momentarily. We describe it in the case $d = 2$. Let R_1 and R_2 be reflection operators that act as follows: for $\mathbf{u} = (u_1, u_2) \in [0,1]^2$

$$R_1(\mathbf{u}) := (1 - u_1, u_2), \qquad R_2(\mathbf{u}) := (u_1, 1 - u_2).$$

For a set $\xi = \{\xi^j\}_{j=1}^N \subset [0,1]^2$, define the symmetrized set

$$\bar{\xi} := \xi \cup R_1(\xi) \cup R_2(\xi) \cup R_2(R_1(\xi)).$$

This set contains $4N$ points, counting multiplicity. The sets

$$G_1(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2} + x_1\right) \times \left[\frac{1}{2}, \frac{1}{2} + x_2\right), \qquad G_2(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2} - x_1\right) \times \left[\frac{1}{2}, \frac{1}{2} + x_2\right),$$

$$G_3(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2} - x_1\right) \times \left[\frac{1}{2}, \frac{1}{2} - x_2\right), \qquad G_4(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2} + x_1\right) \times \left[\frac{1}{2}, \frac{1}{2} - x_2\right),$$

contain the same number of points of $\bar{\xi}$ since we split the points in set $\bar{\xi}$ on the boundary evenly.

We now define ξ^{sym} – the *two-fold symmetrization* of ξ – in the following way: take all the points of $\bar{\xi}$ that lie in the same quadrant $[1/2, 1] \times [1/2, 1]$, then shift and rescale them to the unit square $[0, 1]^2$:

$$\xi^{sym} := \left\{ \mathbf{v} = 2\left(\mathbf{u} - \frac{\mathbf{1}}{2}\right) : \quad \mathbf{u} \in \bar{\xi} \cap \left([\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \right) \right\}. \tag{3.64}$$

Then for the quartered L_p discrepancy of $\bar{\xi}$ we have

$$\begin{aligned} D^q(\bar{\xi}, 4N, 2)_p^p &= 4 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \sum_{\mathbf{u} \in \bar{\xi} \cap [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]} \chi_{G_1(\mathbf{x})}(\mathbf{u}) - 4N \cdot x_1 x_2 \right|^p dx_1 dx_2 \\ &= \int_0^1 \int_0^1 \left| \sum_{\mathbf{v} \in \xi^{sym}} \chi_{[\mathbf{0}, \mathbf{z}]}(\mathbf{v}) - N \cdot z_1 z_2 \right|^p dz_1 dz_2 = \|D(\xi^{sym}, \mathbf{z})\|_p^p, \end{aligned}$$

where $\mathbf{z} = 2\mathbf{x}$. On the other hand, obviously $D^q(\bar{\xi}, 4N, 2)_p = 4D^q(\xi, N, 2)_p$. Thus we have proved the following simple property that we formulate as a proposition.

Proposition 3.1. *Let ξ^{sym} be the two-fold symmetrization of ξ as defined by (3.64). Then*

$$\|D(\xi^{sym}, \mathbf{x})\|_p = 4D^q(\xi, N, 2)_p.$$

Proposition 3.1 can be used in both directions. First, it allows us to get a lower bound for $D^q(\xi, N, 2)_p$. It is known that for all $p > 1$ and any set $\mathcal{P}_N$ of N points one has

$$\|D(\mathcal{P}_N, \mathbf{x})\|_p \geq C\sqrt{\log N}, \tag{3.65}$$

where C is some positive absolute constant. Therefore, for any ξ

$$D^q(\xi, N, 2)_p \geq C\sqrt{\log N}.$$

Second, it gives a way to build a set (in our case ξ^{sym}) with good L_p discrepancy from a set (in our case ξ) with good quartered L_p discrepancy. For instance, as we prove below, the Fibonacci sets $\mathcal{F}_n$ have optimal quartered L_p discrepancy for $p \in (1, \infty)$ in the sense of order. Therefore, by Proposition 3.1 the set $\mathcal{F}_n^{sym}$, obtained from the Fibonacci set $\mathcal{F}_n$ by the symmetrization procedure described above, has optimal in the sense of order standard L_p discrepancy for all $p \in (1, \infty)$.

We proceed to estimate $D^q(\xi,N,d)_p$, $p<\infty$, from above in the case when $\xi=\mathcal{F}_n$ is the Fibonacci set, i.e. $d=2$, $N=b_n$ and

$$\xi^\mu = (\mu/b_n, \{\mu b_{n-1}/b_n\}), \qquad \xi = \mathcal{F}_n := \{\xi^\mu\}_{\mu=1}^{b_n}.$$

We apply the technique that is based on the Fourier representation of $S(\mathbf{x},\mathbf{y})$ as a function on $\mathbf{y}$. First, we find the Fourier coefficients of the univariate function

$$\hat{S}(a,k) = \int_0^1 S(a,t)e^{-2\pi ikt}dt = (-1)^k(2\pi ik)^{-1}(e^{2\pi ika}-e^{-2\pi ika}).$$

It is clear that $\hat{S}(a,0)=2a$. Second, it follows directly from the definition of $S(\mathbf{x},\mathbf{y})$ and the above formulas that

$$|\hat{S}(\mathbf{x},\mathbf{k})| = \prod_{j=1}^d |\hat{S}(x_j,k_j)| \le \prod_{j=1}^d \max(|k_j|,1)^{-1}. \tag{3.66}$$

Denote

$$\Phi(\mathbf{k}) = \sum_{\mu=1}^{b_n} e^{2\pi i(\mathbf{k},\xi^\mu)}.$$

Then for a trigonometric polynomial f one has

$$\Phi_n(f) := \sum_{\mu=1}^{b_n} f\big(\mu/b_n, \{\mu b_{n-1}/b_n\}\big) = \sum_{\mathbf{k}} \hat{f}(\mathbf{k})\Phi(\mathbf{k}). \tag{3.67}$$

It is known and easy to see that the following relation holds

$$\Phi(\mathbf{k}) = \begin{cases} b_n, & \mathbf{k}\in L(n), \\ 0, & \mathbf{k}\notin L(n). \end{cases} \tag{3.68}$$

Therefore, in the case $p=2$, that we discuss first

$$D^q(\mathcal{F}_n,b_n,2)_2 \le \left\| \sum_{\mathbf{k}\ne(0,0)} \Phi(\mathbf{k})\hat{S}(\mathbf{x},\mathbf{k}) \right\|_2. \tag{3.69}$$

Using the fact that functions $\hat{S}(\mathbf{x},\mathbf{k})$ and $\hat{S}(\mathbf{x},\mathbf{k}')$ are orthogonal on $[0,1]^2$ if $(|k_1|,|k_2|)\ne(|k_1'|,|k_2'|)$, the bounds (3.66), (3.69), and estimate (2.51) we obtain

$$D^q(\mathcal{F}_n,b_n,2)_2^2 \ll \sum_{l=0}^\infty b_n^2(2^lb_n)^{-2}|Z_l| \ll \log b_n \sum_{l=0}^\infty \frac{2^l(l+1)}{2^{2l}} \ll \log b_n. \tag{3.70}$$

Thus,

$$D^q(\mathcal{F}_n, b_n, 2)_2 \ll \sqrt{\log b_n}. \tag{3.71}$$

We now proceed to the case $p \in [2, \infty)$. Let

$$\psi_l(\mathbf{x}) := \sum_{k \in Z_l} \hat{S}(\mathbf{x}, \mathbf{k}).$$

Then

$$D^q(\mathcal{F}_n, b_n, 2)_p \le b_n \sum_{l=0}^{\infty} \|\psi_l\|_p. \tag{3.72}$$

By the corollary of the Littlewood-Paley theorem we have for $\|\psi_l\|_p$

$$\|\psi_l\|_p \ll \left(\sum_{\mathbf{s}} \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_p^2 \right)^{1/2}, \tag{3.73}$$

where for $\mathbf{s} = (s_1, s_2)$, s_j are nonnegative integers

$$\delta_{\mathbf{s}}(f, \mathbf{x}) := \sum_{\substack{[2^{s_j - 1}] \le |k_j| < 2^{s_j}, \\ j=1,2}} \hat{f}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{x})}.$$

It is not difficult to see that for ψ_l only those $\delta_{\mathbf{s}}(\psi_l)$ can be nonzero for which

$$\big| \, \|\mathbf{s}\|_1 - \log_2(2^l \gamma b_n) \big| \le C.$$

In addition by Lemma 2.9 the number of terms of $\delta_{\mathbf{s}}(\psi_l)$ is not greater than $C2^l$. Therefore,

$$\big\| \delta_{\mathbf{s}}(\psi_l) \big\|_p \le \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_2^{2/p} \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_\infty^{1-2/p} \ll 2^{-l/p} b_n^{-1} \tag{3.74}$$

and

$$\|\psi_l\|_p \ll (l + \log b_n)^{1/2} 2^{-l/p} b_n^{-1}. \tag{3.75}$$

The bounds (3.72), (3.74) and Proposition 3.1 imply

Theorem 3.2. *i) For all $p \in (1, \infty)$, the quartered L_p discrepancy of the Fibonacci set $\mathcal{F}_n$ satisfies*

$$D^q(\mathcal{F}_n, b_n, 2)_p \le C(p) \sqrt{\log b_n}. \tag{3.76}$$

ii) For all $p \in (1, \infty)$, the two-fold symmetrization $\mathcal{F}_n^{sym}$ of the Fibonacci set $\mathcal{F}_n$ has optimal L_p discrepancy:

$$\|D(\mathcal{F}_n^{sym}, \mathbf{x})\|_p \le C'(p) \sqrt{\log 4b_n}. \tag{3.77}$$

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