On some sets with minimal L^2 discrepancy

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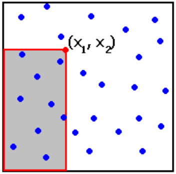
Discrepancy function

Consider $\mathcal{P}_N \subset [0,1]^d$ with $\#\mathcal{P}_N = N$:



$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - Nx_1x_2\ldots x_d$$

L^∞ estimates (star-discrepancy)

Theorem (Schmidt, 1972)

For $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$

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There exist $\mathcal{P}_N \subset [0,1]^2$ with $\|D_N\|_\infty \approx \log N$

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There exist $\mathcal{P}_N \subset [0,1]^d$ with $\|D_N\|_\infty \lesssim (\log N)^{d-1}$

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Theorem (DB, Lacey, Vagharshakyan, 2007)

For $d \geq 3$ there exists $0 < \varepsilon_d \leq \frac{1}{2}$, such that
$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\varepsilon_d}$

L^p estimates, $1 < p < \infty$

Theorem

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

Roth ($p = 2$) 1954, Schmidt ($p \neq 2$) 1977

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Theorem

There exist $\mathcal{P}_N \subset [0,1]^d$ with

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(Davenport; Roth; Halton, Zaremba; Chen, Skriganov)

Van der Corput set

"Digit reversing" van der Corput set

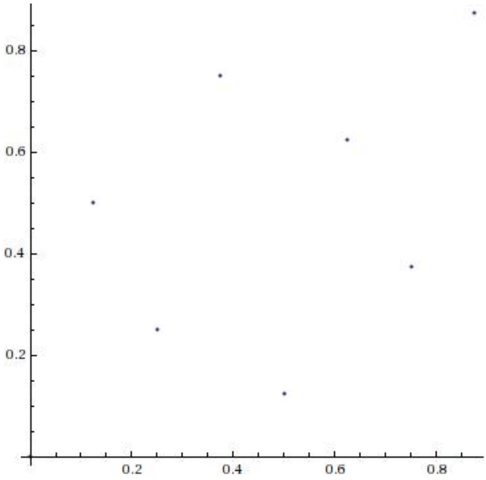
Denote the binary expansion of $x \in [0,1)$ by

$$x = \sum_i x_i \cdot 2^{-i} = 0.x_1 x_2 ... x_n ...$$

The van der Corput set $\mathcal{V}_n$ with 2^n points is defined as:

$$\mathcal{V}_n = \{ \, (\, 0.x_1 x_2 \ldots x_{n-1} x_n \, , \, 0.x_n x_{n-1} \ldots x_2 x_1 \,) \, : \, x_i = 0, 1 \}$$

van der Corput set



van der Corput set with $N = 2^3$ points

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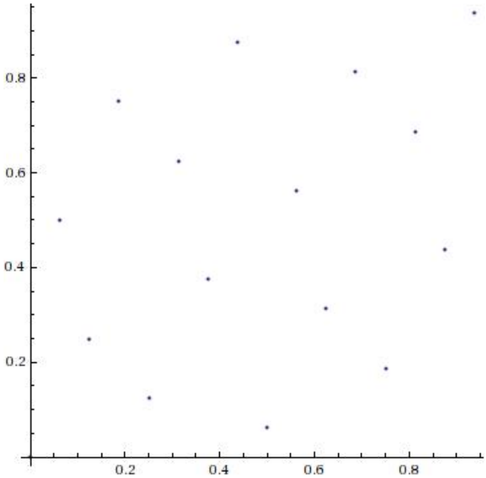
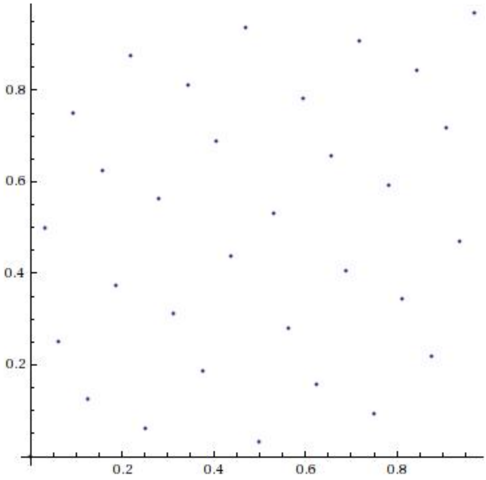


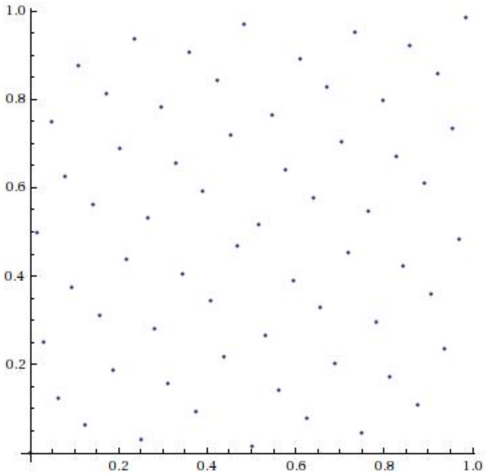
van der Corput set with $N = 2^4$ points

van der Corput set



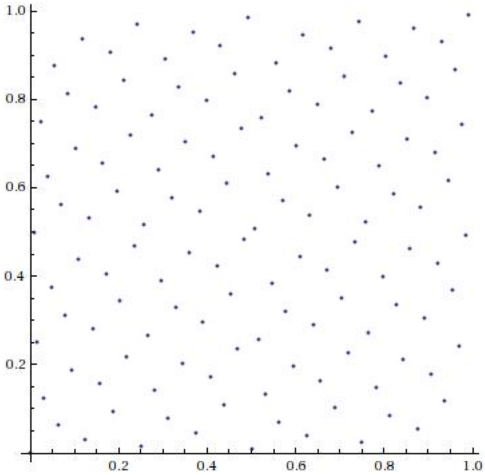
van der Corput set with $N = 2^5$ points

van der Corput set



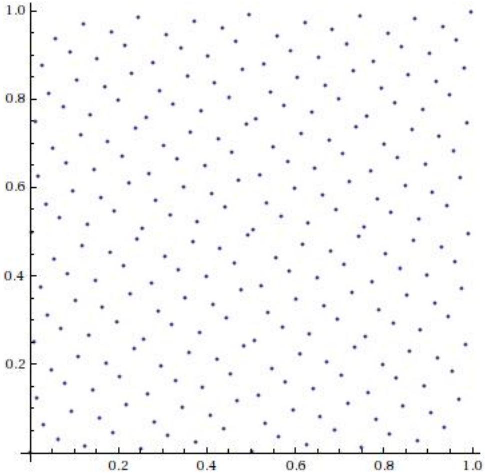
van der Corput set with $N = 2^6$ points

van der Corput set



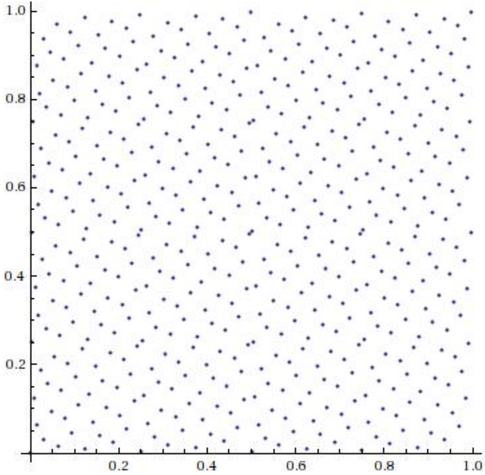
van der Corput set with $N = 2^7$ points

van der Corput set



van der Corput set with $N = 2^8$ points

van der Corput set



van der Corput set with $N = 2^9$ points

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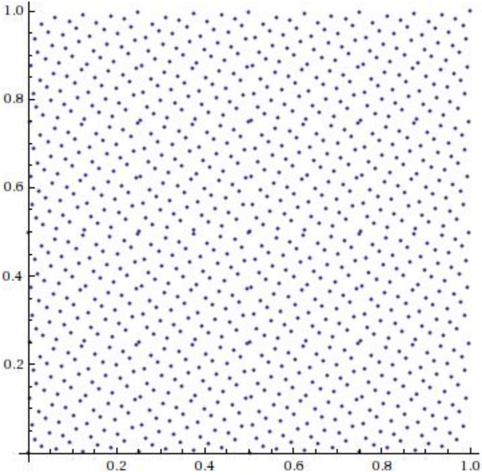
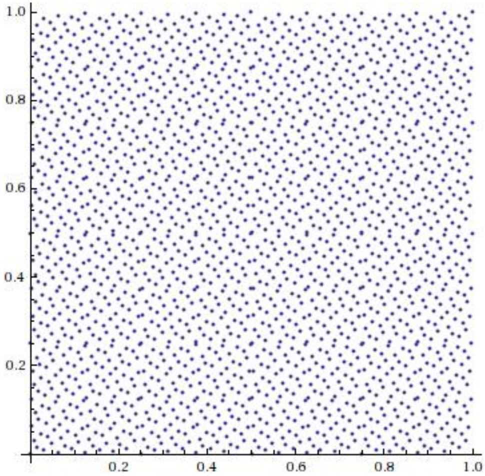


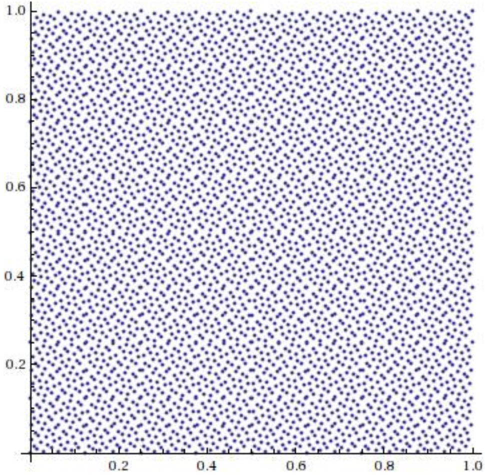
van der Corput set with $N = 2^{10}$ points

van der Corput set



van der Corput set with $N = 2^{11}$ points

van der Corput set



van der Corput set with $N = 2^{12}$ points

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Theorem (van der Corput)

The set $\mathcal{V}_n$ satisfies with $\|D_{\mathcal{V}_n}\|_\infty \lesssim n \approx \log N$

Irrational lattice

Example

Let α be an irrational number and let $\{x\}$ denote the fractional part of x.

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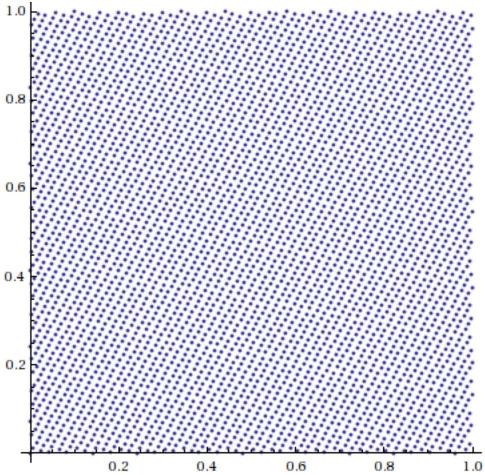
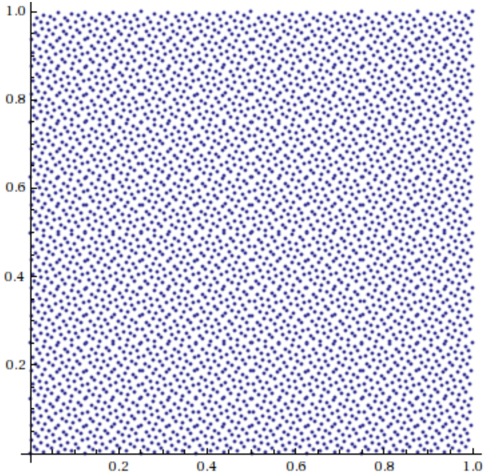
- The idea goes as far back as 1904 (Lerch)

Low discrepancy sets



The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points
Discrepancy $\approx \log N$

Low discrepancy sets



The van der Corput set with $N = 2^{12}$ points

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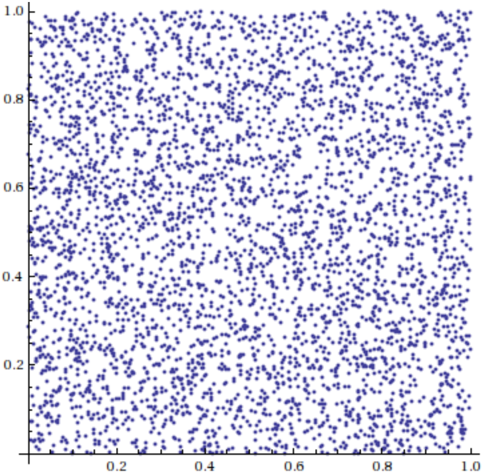
Low discrepancy sets



Random set with $N = 2^{12}$ points
Discrepancy $\approx \sqrt{N}$

L^2 bounds

Theorem (K. Roth)

In dimension $d = 2$, for any N-point set $\mathcal{P}_N \subset [0,1]^2$,

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Standard sets fail to meet this bound

For the van der Corput set and the irrational lattice, we have

$$\|D_N\|_2 \approx \log N$$

Remedies

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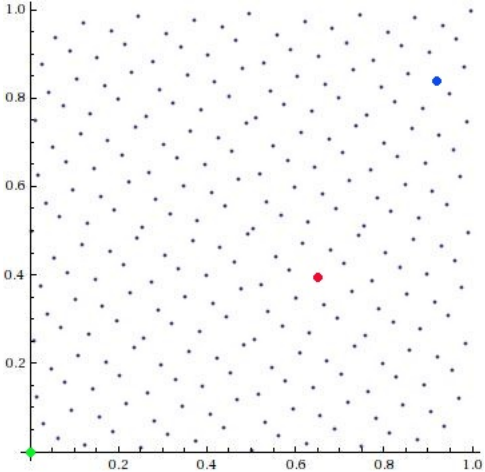
Remedy: Cyclic shifts

Define $\mathcal{V}_n^\alpha = \left\{ \big((x+\alpha) \bmod 1, y \big) : (x,y) \in \mathcal{V}_n \right\}$



van der Corput set with $N = 2^8$ points

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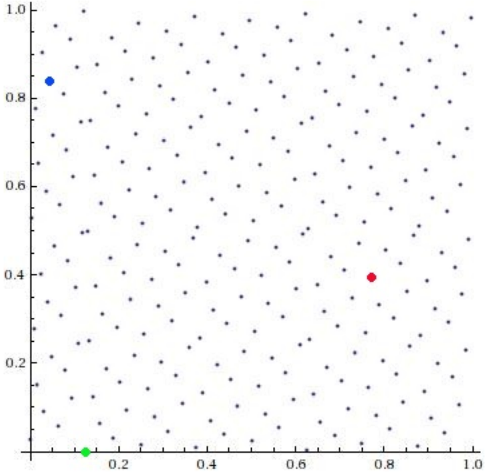

van der Corput set with $N = 2^8$ points
translated (mod 1) by 1/8

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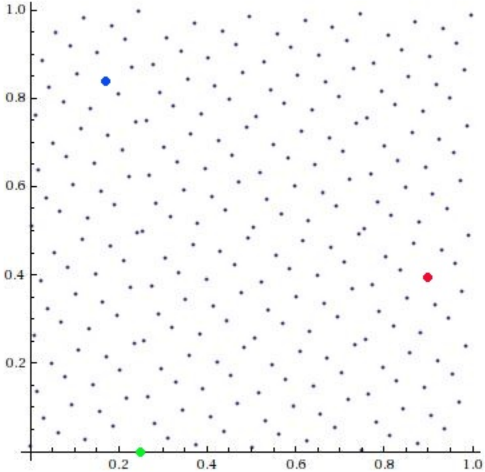

van der Corput set with $N = 2^8$ points
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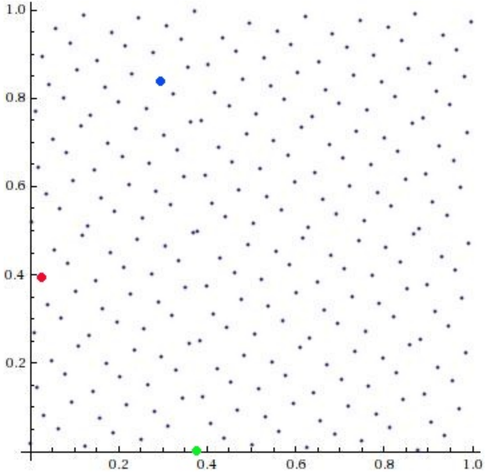

van der Corput set with $N = 2^8$ points
translated (mod 1) by 3/8

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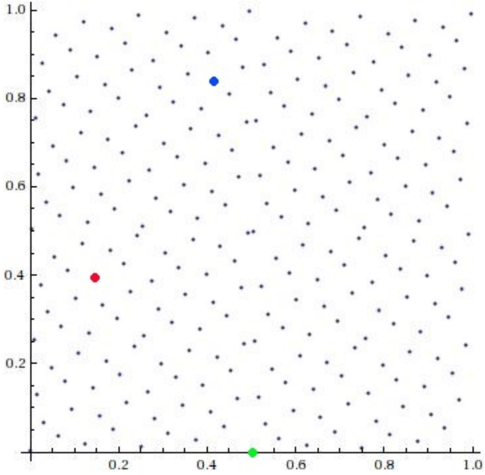

van der Corput set with $N = 2^8$ points
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Theorem (K. Roth, 1979)

$$\mathbb{E}_\alpha \big\| D_{\mathcal{V}_n^\alpha} \big\|_2 \lesssim \sqrt{\log N}$$

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Theorem (K. Roth, 1979)

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Theorem (D.B., 2008)

For $\alpha = 1 - \frac{k}{2^n}$, where

$$k = \Big(\underbrace{000111\ldots000111}_{n_1 \text{ digits}} \underbrace{00001111\ldots00001111}_{n_2 \text{ digits}} \Big)_2 \qquad \frac{n_1}{n_2} = \frac{54}{17}$$

we have

$$\big\| D_{\mathcal{V}_n^\alpha} \big\|_2 \lesssim \sqrt{\log N}$$

The integral

The integral of discrepancy is big

$$\int_0^1 \int_0^1 D_{\mathcal{V}_n}(x)dx = \frac{n}{8} + \mathcal{O}(1)$$

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$$= 2^n \mathbb{E}\Big(1 - \sum_{k=1}^{n} X_k \cdot 2^{-k}\Big) \cdot \Big(1 - \sum_{j=1}^{n} X_j \cdot 2^{-n+j-1}\Big)$$

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where X_i are i.i.d with $Pr(X_i = 0) = Pr(X_i) = \frac{1}{2}$
$\mathbb{E}X_i \cdot X_j = \frac{1}{4}$, but $\mathbb{E}X_j^2 = \frac{1}{2}$

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Theorem (Halton and Zaremba, 1968)

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Theorem (Halton and Zaremba, 1968)

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$$\int D_{\mathcal{V}_n^\alpha} \approx \int D_{\mathcal{V}_n} - \frac{k}{2} + \sum_{p \in \mathcal{V}_n \colon p_1 \leq k/2^n} p_2$$

Proof

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$$f_l(k) = \begin{cases} 2^{l-1} m & \text{if} \quad k = 2^l m + r, \qquad 0 \le r < 2^{l-1} \\ 2^{l-1} m + 1 + r & \text{if} \quad k = 2^l m + 2^{l-1} + r, \; 0 \le r < 2^{l-1} \end{cases}$$

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$$\sum_{p_1\leq k/2^n} p_2 = \sum_{l=1}^{n-1} \sum_{i=l+1}^{n} k_i \cdot 2^{i-l-2} + \sum_{l=1}^{n} k_l \cdot 2^{-l} + \sum_{l=2}^{n} \sum_{i=1}^{l-1} k_i \cdot k_l \cdot 2^{i-l-1}$$

$$\approx \frac{1}{2}\Big(k - \sum k_i\Big)$$

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$$f_l(k) = \sum_{i=l+1}^{n} k_i \cdot 2^{i-2} + k_l \cdot \sum_{i=1}^{l-1} k_i \cdot 2^{i-1} + k_l$$

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We want:

$$\sum_{i=1}^{n} k_i = \frac{n}{2} + \mathcal{O}(1) \qquad \text{and} \qquad \mathcal{S} = \sum_{l=2}^{n} \sum_{i=1}^{l-1} k_i \cdot k_l \cdot 2^{i-l-1} = \frac{n}{8} + \mathcal{O}(1)$$

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- $k = (00001111\ldots00001111)_2$: $\mathcal{S} \approx \frac{19}{136} n$

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We want:

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- $k = \Big(\underbrace{000111 \ldots 000111}_{n_1 \text{ digits}} \underbrace{00001111 \ldots 00001111}_{n_2 \text{ digits}}\Big)_2 \qquad \frac{n_1}{n_2} = \frac{54}{17}$

$$\mathcal{S} = \frac{n}{8} + \mathcal{O}(1)$$

Fourier coefficients $\widehat{D_{\mathcal{V}_n^\alpha}}(n_1, n_2)$: $(n_1, n_2) \neq 0$

- $n_1 \neq 0$ OR $n_1 = 0,\ n_2 \equiv 0 \mod 2^n$

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- Thus,

$$\left\| \left(D_{\mathcal{V}_n^{\alpha_0}} - D_{\mathcal{V}_n} \right)_{\{n_1=0, n_2\neq 0\}} \right\|_2^2 \lesssim \sum_{s=0}^{n-1} \sum_{m \text{ odd}} \frac{1}{m^2} \lesssim n = \log N$$

Davenport's reflection principle

Remedy: Davenport's reflection

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- $\mathcal{P}_N \longrightarrow \mathcal{P} \cup \{(1-x,y) : (x,y) \in \mathcal{P}_N\}$

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- Irrational lattice: Davenport (1956)
- van der Corput set: Chen, Skriganov (2003)

Fibonacci set

Example

Let $\{b_n\}_{n=1}^{\infty}$ be the Fibonacci numbers.

Define $\mathcal{F}_n = \left\{ \left(\frac{k}{b_n}, \left\{ k \frac{b_{n-1}}{b_n} \right\} \right) \right\}_{k=0}^{b_n - 1}$

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- Define the symmetrization of $\mathcal{F}_n$:
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Theorem (DB, V.Temlyakov, R.Yu)

For a symmetrized set $\mathcal{F}'_n$

$$\|D_{\mathcal{F}'_n}\|_2 \approx (\log N)^{1/2}.$$

Fibonacci set

Theorem (DB, Temlyakov, Yu)

$$\|D_{\mathcal{F}'_n}\|_2^2 = \frac{1}{8b_n^2}\sum_{r=1}^{b_n-1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right)\cdot \sin^2\left(\frac{\pi r}{b_n}\right)} + \frac{17}{36} - \frac{1}{36 b_n^2}$$

when b_n is odd,

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Fibonacci set

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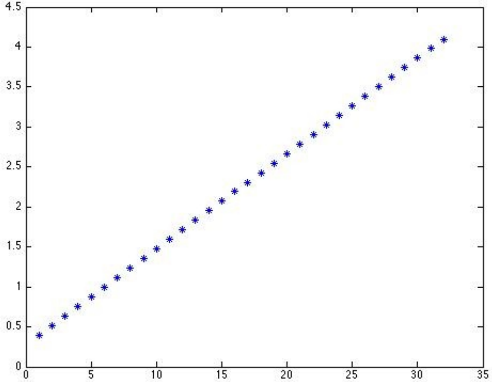

Remedy: Digital shift

Digital shift

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Digital shift

- Define

$$\mathcal{V}_{n,\sigma} = \Big\{ \Big(0.x_1 x_2 \ldots x_{n-1} x_n \,,\, 0.(x_n \oplus \sigma_n) \ldots (x_2 \oplus \sigma_2)(x_1 \oplus \sigma_1) \Big) \Big\}$$

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- Chen & Skriganov; Niederreiter; Pillichshammer, Larcher, Faure, Kritzer, etc

$\exp(L^\alpha)$ estimates: B., Lacey, Parissis, Vagharshakyan

Theorem (DB, Lacey, Parissis, Vagharshakyan 2008)

For any N-point set $\mathcal{P}_N \subset [0,1]^2$ we have

$$\|D_N\|_{\exp(L^\alpha)} \gtrsim (\log N)^{1-1/\alpha}, \qquad 2 \le \alpha < \infty .$$

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For $1 < p < \infty$, the digit-shifted van der Corput set satisfies

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BMO estimates: B., Lacey, Parissis, Vagharshakyan

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$$\|f\|_{\mathrm{BMO}} := \sup_{U \subset [0,1]^2} \left[|U|^{-1} \sum_{\substack{R \in \mathcal{D}^2 \\ R \subset U}} \frac{\langle f, h_R \rangle^2}{|R|} \right]^{1/2} < \infty .$$

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