A Primer on Implied Correlation

John A. Dodson*

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This note summarizes a presentation delivered to the University of Minnesota Financial Mathematics Practitioner Seminar on August 15, 2007. The theme for the summer session was credit derivatives, and in particular a reading of Philipp Schönbucher's text, *Credit derivative pricing models - Models, Pricing, and Implementation*, published by Wiley Finance in 2003.

1 Indexes

Index publishers have started to create standard portfolios of CDS's which are becoming the basis for structured securities issued by investment banks. The two main services are CDX in the U.S. and iTraxx in Europe. The CDX is published by Dow Jones. The investment-grade (IG) portfolio consists of equal-weighted positions in CDS's in 125 obligors deemed investment grade at inception. New 5-, 7-, and 10-year portfolios are formed twice per year. The constituent CDS's in each are priced every day and the average rate is reported.

Standard tranchings of the claims against this portfolio have been developed. These were originally designed to garner specific ratings. For our purposes, the tranches are defined by loss attachment and detachment points.

equity	0 % - 3 %
junior mezzanine	3 % - 7 %
senior mezzanine	7 % - 10 %
senior	10 % - 15 %
junior super senior	15 % - 30 %
super senior	30 % - 100 %

Holders of securities based on these tranches can expect to receive permanent losses in the event of claims against the CDS's if and only if lower tranches have received permanent losses.

The financial engineering is achieved by setting up a trust funded by the issuance of certificates, 3 % as equity, 70 % as super senior notes, 15 % as junior super senior notes, etc. These proceeds are used as collateral against which to write 125 credit

*University of Minnesota Financial Mathematics program & RiverSource Investments investment risk management

default swaps. The income from these swaps and interest on the collateral is used to pay the certificate coupons. Any successful claims by the CDS buyers is paid out of the collateral; and the remainder, net of recovery proceeds, is paid to certificate holders by priority at maturity.

There seems to be some ambiguity, in my mind at least, about the treatment of recovery proceeds. The Morgan Stanley reference I have says that, for the equity tranche at least, the detachment refers to total portfolio losses. So the value of the equity tranche can go to zero. For higher tranches, it is not clear whether recoveries are segmented to loss layers or whether their ultimate recoveries, too, can potentially be zero.

2 Standard Model

In analogy to the Black-Scholes formula for options, we make no representation about the validity of assumptions behind the standard model. Rather, the model can be thought of as a basis for the quoting convention.

2.1 Assumptions

- homogeneous $\mathrm{Bern}(p)$ partials on the risk-neutral default events over the term
- dependencies described by a normal copula
- ...with constant pair-wise correlation, ρ
- constant recovery rate $1-L$ on CDS claims
- large-portfolio limit

3 Loss Distribution

From the text, (10.25), we know that in the large-portfolio limit, the distribution of the fraction of CDS's that receive default claims is

$$F_X(x|p,\rho) = \Phi\left(\sqrt{\frac{1}{\rho}-1}\cdot\Phi^{-1}(p) - \sqrt{\frac{1}{\rho}}\cdot\Phi^{-1}(x)\right) \tag{3.1}$$

for $0 \leq x \leq 1$, where

$$\Phi(z) = \frac{1}{2}\cdot\left(1+\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \tag{3.2}$$

is the standard normal CDF.

The ultimate loss is $Y = L\cdot X$, so the loss distribution under the assumptions is

$$F_Y(y|L,p,\rho) = \frac{1}{L}\cdot\Phi\left(\sqrt{\frac{1}{\rho}-1}\cdot\Phi^{-1}(p) - \sqrt{\frac{1}{\rho}}\cdot\Phi^{-1}\left(\frac{y}{L}\right)\right) \tag{3.3}$$

for $0 \leq y \leq L$. This can be used to price the portfolio and the tranches.

4 Calibration

Our aim is to use the standard model to fit ρ to tranche spreads. Just as we use the term "implied volatility" in the Black-Scholes setting, we will call this "compound correlation."

4.1 Recovery

Clearly the actual nature of default recovery—both the individual CDS recoveries and how the ultimate portfolio value is distributed to CDO holders—is very important to a tranche holder. For the purposes of the standard model, it is conventional to use $1 - L = 0.4$ or $L = 0.6$. Note that $L = 0.6$ means that no tranche except equity experiences a loss as long as the portfolio default rate does not exceed 5 %.

This heuristic is based on historical recovery rates for senior un-secured bonds according to rating agencies.

Note that the super senior tranche is risk-free if $L < 0.3$.

Additionally, let us assume that as long as $L < 0.7$ all tranches except the super senior can potentially have zero recovery.

4.2 Marginal risk-neutral default probability

Since the spread for each CDS in the portfolio is readily available, it would seem that one could use a different parameter for each bernoulli marginal. In practice, it seems that the tranche valuation is not very sensitive to the manner in which the portfolio's constituents are modeled. The homogenous, large-portfolio limit is reasonable for pricing newly-issued tranches, and therefore for the derivation of implied correlations.

Since $\mathsf{E}X = p$, the risk-neutral expected principle loss on the whole portfolio is $L \cdot p$. If we assume that the recovery is net of the scheduled coupon payments, then we can use the CDX rate for the portfolio, indicated by $\bar{s}(T)$ since it is an average of CDS rates, to solve for p.

Working through the bond math as we do in the appendix, we get

$$p = \frac{\bar{s}(T)}{L} \cdot \frac{1}{s(T)} \cdot \left(\frac{1}{B(t)} - 1 \right) \tag{4.1}$$

where $s(T)$ is the interbank swap rate and $B(T)$ is the risk-free discount factor in the notation of the text.

Note that for T small we get

$$p \approx \frac{\bar{s}(T)}{L} \cdot T \tag{4.2}$$

which is reminiscent of the poisson process.

5 Compound Correlation

Similarly, if we have a spread $\bar{s}(T, K_0, K_1)$ for a CDO tranche with attachment $K_0 \geq 0$ and detachment $K_1 > K_0$, we can work out that risk-neutral expected loss is defined by

$$\frac{\bar{s}(T,K_0,K_1)}{s(T)} \cdot \left(\frac{1}{B(T)} - 1\right) = \frac{1}{K_1 - K_0} \cdot \int_{K_0 \wedge L}^{K_1 \wedge L} y \cdot F_Y' \left(y|L, p, \rho_{K_0}(K_1)\right) \, dy \quad (5.1)$$

where $\rho_{K_0}(K_1)$ is the compound correlation for the tranche.

5.1 A problem

It would be satisfying to know that the compound correlation is uniquely defined by (5.1). This is unfortunately not the case.

Some intuition comes from a consideration of the sensitivity of the tranche value to the level of implied correlation. Intuitively, it would seem that holders of higher quality tranches would prefer lower correlations so that losses are spread out and less likely to enter their layer. On the other hand, equity and possibly mezzanine tranches would prefer higher correlations so that any losses are borne deep into the capital structure. This suggests that there may be both low and high correlation solutions to (5.1) for some tranches; and in fact this is the case.

6 Base Correlation

The solution to this problem is to consider only synthetic equity tranches with attachment zero. The spread on a synthetic equity tranche with detachment K must be

$$\bar{s}(T,0,K) = \frac{1}{K} \cdot \sum_{i=1}^{n} (K_i - K_{i-1}) \cdot \bar{s}(T, K_{i-1}, K_i) \quad (6.1)$$

for $K_0 = 0$, $K_n = K$, and $K_i > K_{i-1}$.

The contention, herein left unproven, is that

$$\int_0^{K \wedge L} y \cdot F_Y' \left(y|L, p, \rho\right) \, dy \quad (6.2)$$

is monotonic in ρ; and therefore there is at most one solution, $\rho_0(K)$, to (5.1) for a given $\bar{s}(T,0,K)$.

The quantity $\rho_0(K)$ is termed the "base correlation."

Newly-issued CDO's based on the CDX are quoted by market makers in terms of the base correlations for the attachment and detachment points of the tranche.

7 Discussion

Were the assumptions of the standard model valid, the base correlation would not depend on the detachment point. In practice, $\rho_0(K)$ seems to be upward sloping in K. This would seem to be consistent with a contagion-style model.

A Appendix: CDO spread

Let us discuss the result (4.1),

$$p \cdot L = \frac{\bar{s}(T)}{s(T)} \cdot \left(\frac{1}{B(T)} - 1 \right) \tag{A.1}$$

for the inferred loss on a CDO.

How the timing and recovery of CDS claims is modeled clearly influences the valuation of CDO's. Furthermore, frictions such as structuring and management fees that separate the value of CDO's from the value of the portfolio may be material. Nonetheless, we believe (4.1) is reasonable for the purposes of the standard model.

In this appendix, we would like to demonstrate the derivation in case it is not obvious.

A.1 Interbank swap rates

Under continuous compounding, the fixed-floating no-arbitrage relationship provides

$$1 - B(T) = s(T) \cdot \int_0^T B(t)\, dt \tag{A.2}$$

where $B(t)$ is the term structure of risk-free discount factors and $s(T)$ is the swap rate for term T.

A.2 Tranche spread

Generally, CDO tranche securities pay a floating coupon at a fixed spread to the interbank rate. At inception, this is equivalent to a fixed coupon equal to the sum of the swap rate and the spread, $s(T) + \bar{s}(T)$.

Let us assume that the ultimate portfolio losses from CDS claims less recovery is insufficient to interfere with the payment of all scheduled coupons, and that the loss is distributed through the capital structure only at maturity. Furthermore, let us interpet the loss rate per claim, L, as a forward value.

The ultimate loss, Y, is a random variable. In order for the claim to be initially valued at par, no-arbitrage requires

$$1 = (s(T) + \bar{s}(T)) \cdot \int_0^T B(t)\, dt + (1 - \mathsf{E}Y) \cdot B(T) \tag{A.3}$$

$$= (s(T) + \bar{s}(T)) \cdot \frac{1 - B(T)}{s(T)} + (1 - L \cdot p) \cdot B(T) \tag{A.4}$$

where $\mathsf{E}Y$ under the risk-neutral measure is $L \cdot p$ for the entire portfolio.

The result follows by re-arrangement.