A General Approach to Portfolio Risk Measurement

by

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Portfolio Risk Measurement

- The economic risk / return trade-off
  - Theoretical & Practice challenges
- Measurement vs. Management
  - Control vs. Trading
- Standards & Regulations
  - Generally Accepted Principals, disclosure

Risk Classifications

- *Bancassurance*: Banking & Insurance
  - Banking
    - Market Risk
    - Credit Risk
    - Liquidity Risk
    - Operational Risk
  - Insurance
    - Property / Casualty …

Market Risk

- On-balance sheet
  - Assets
  - Liabilities
- Off-balance sheet
  - Reserves
  - Translations
  - Hedges

Brownian Motion

- Continuous
- Finite quadratic variation
- Normal increments

$$dz \equiv dB(t) \qquad dz \cdot dz = dt$$

$$B(t+s) - B(t) \sim \mathrm{N}(0, s^{+})$$

Market Factors

- Factors may or may not be prices
- Integrands are general non-stochastic functions of factor levels and time

$$dx^i = M^i\left(t, x^k\right)dt + L^i{}_j\left(t, x^k\right)dz^j$$

$$dx^i \cdot dx_j = L^i{}_k \cdot L_j{}^k\, dt \equiv \Sigma^i_j\, dt$$

Example From Cholesky Decomposition

- Two-factor correlated geometric Brownian motion

$$\begin{pmatrix} M^1 \\ M^2 \end{pmatrix} = \begin{pmatrix} x^1 \cdot \mu^1 \\ x^2 \cdot \mu^2 \end{pmatrix}$$

$$\begin{pmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{pmatrix} = \begin{pmatrix} x^1 \cdot \sigma^1 & 0 \\ x^2 \cdot \sigma^2 \cdot \rho & x^2 \cdot \sigma^2 \cdot \sqrt{1-\rho^2} \end{pmatrix}$$

Portfolio

- Value depends on factor levels and time
- Assumption: no cash flows, differentiable

$$\Pi\left(t, x^i\right)$$

Portfolio Process

- From Itô's lemma, we can write down the process for the portfolio value

$$d\Pi = \frac{\partial \Pi}{\partial t} dt + \frac{\partial \Pi}{\partial x^i} dx^i + \tfrac{1}{2} \cdot \frac{\partial^2 \Pi}{\partial x^i \, \partial x_j} dx^i \cdot dx_j$$

Portfolio Partials

- Observable parameters
  - Time decay, position, & convexity

$$\frac{\partial \Pi}{\partial t} \equiv \Theta \qquad \frac{\partial \Pi}{\partial x^i} \equiv \Delta_i \qquad \frac{\partial^2 \Pi}{\partial x^i \, \partial x_j} \equiv \Gamma_i^{\ j}$$

$$d\Pi = \Theta \, dt + \Delta_i \cdot \left(M^i \, dt + L^i_{\ j} \, dz^j \right) + \tfrac{1}{2} \Gamma_i^{\ j} \cdot \Sigma^i_j \, dt$$

Portfolio Arbitrage

- *Ansatz*: There exists a hedging portfolio
  - Same delta
  - Different theta and gamma (could be zero)

$$d\Pi' = \Theta' \, dt + \Delta_i \cdot \left(M^i \, dt + L^i{}_j \, dz^j \right) + \tfrac{1}{2} \cdot \Gamma_i'^j \cdot \Sigma^i_j \, dt$$

Portfolio Arbitrage

- The hedged portfolio is risk-free

$$
\begin{aligned}
d\left(\Pi' - \Pi\right) &= \left(\Theta' + \tfrac{1}{2}\Gamma_i'^{\,j} \cdot \Sigma_j^i\right)dt - \left(\Theta + \tfrac{1}{2}\Gamma_i^{\,j} \cdot \Sigma_j^i\right)dt \\
&= \left(\Pi' - \Pi\right) \cdot r\, dt
\end{aligned}
$$

Factor Price

- The resulting operator is proportional to delta
  - Implicit function theorem for linear systems
- Defines an arbitrage constraint

$$\Pi \cdot r - \Theta - \tfrac{1}{2}\Gamma_i^{\ j} \cdot \Sigma_j^i \equiv \Delta_{\ k} \cdot S^k \cdot r$$

Portfolio Process

- Insert the arbitrage constraint into the portfolio process to eliminate gamma & theta

$$d\Pi = \left\{\Pi \cdot r + \Delta_i \cdot \left(M^i - S^i \cdot r\right)\right\} dt + \Delta_i \cdot L^i{}_j \, dz^j$$

Carry Adjustment

- We are led to define a new portfolio consisting of fully-leveraged positions

$$\tilde{M}^i \equiv M^i - S^i \cdot r$$

$$d\tilde{\Pi} \equiv d\Pi - \Pi \cdot r\,dt$$

$$d\tilde{\Pi} = \Delta_i \cdot \tilde{M}^i\,dt + \Delta_i \cdot L^i{}_j\,dz^j$$

Risk As Quadratic Variation

- Instantaneous variance of portfolio profit/loss

$$
\begin{aligned}
d\Pi \cdot d\Pi &= \Delta_i \cdot L^i{}_k \cdot L_j{}^k \cdot \Delta^j \, dt \\
&= \Delta_i \cdot \Sigma^i{}_j \cdot \Delta^j \, dt \\
&\equiv V^2 \, dt
\end{aligned}
$$

Marginal Risk

- The incremental risk from a given position
  - The basis for decision making
- Additive decomposition of total risk

$$N^i \equiv \frac{\partial V}{\partial \Delta_i} = \frac{\Sigma^i_j \cdot \Delta^j}{V}$$

$$V \equiv \sqrt{\Delta_i \cdot \Sigma^i_j \cdot \Delta^j} = \Delta_i \cdot N^i$$

Marginal Risk

- Project portfolio onto a single stochastic component

$$d\tilde{\Pi} = \Delta_i \cdot \tilde{M}^i \, dt + \Delta_i \cdot N^i \, dz$$

Efficiency

- The marginal return should be proportional to the marginal risk
  - Maximum return for a given level of risk

$$\tilde{N}^i \equiv \lambda \cdot \tilde{M}^i$$

$$\lambda \equiv \sqrt{\tilde{M}_j \cdot \mathrm{T}_i^j \cdot \tilde{M}^i}$$

$$\mathrm{T} \equiv \Sigma^{-1}$$

Efficiency

- This has various interpretations
  - Unique market price of risk
  - Slope of the Capital Market Line

$$d\tilde{\Pi}_{eff} = V \cdot \lambda \, dt + V \, dz$$

$$\lambda = \frac{\mu - r}{\sigma}$$

Finite Increments

- Over a finite time horizon, the portfolio profit/loss is given by a stochastic integral

$$\delta\Pi \equiv \int_0^{\delta t} d\Pi$$

$$= \int_0^{\delta t} \left(\Pi \cdot r + \Delta_i \cdot \tilde{M}^i\right) dt + \int_0^{\delta t} \Delta_i \cdot L^i{}_j \, dz^j$$

Finite Increments

- The stochastic term generally dominates the non-stochastic term for short time intervals
- Allow it to vary with respect to factor levels
  - *e.g.* Geometric Brownian motion, trading patterns

$$\Delta_i\left(x^k\right)\cdot L^i{}_j\left(x^k\right)$$

Linear Approximation

- First-order Taylor's expansion

$$\Delta_i \cdot L^i{}_j \approx \Delta_i\big|_0 \cdot L^i{}_j\big|_0 + \left(\frac{\partial \Delta_i}{\partial x_k}\bigg|_0 \cdot L^i{}_j\big|_0 + \Delta_i\big|_0 \cdot \frac{\partial L^i{}_j}{\partial x_k}\bigg|_0 \right) \cdot \frac{\partial x_k}{\partial z_l}\bigg|_0 \cdot z_l$$

$$= \Delta_i\big|_0 \cdot L^i{}_j\big|_0 + \left(\Gamma_i{}^k\big|_0 \cdot L^i{}_j\big|_0 + \Delta_i\big|_0 \cdot L^i{}_j{}^{,k}\big|_0 \right) \cdot L_k{}^l\big|_0 \cdot z_l$$

Stochastic Integration Results

- Integrate one Brownian motion w.r.t. another

$$\int_0^{\delta t} dz^j = z^j \cdot \sqrt{\delta t} \quad , \quad z^j \sim \text{iid}\, \mathrm{N}(0,1)$$

$$\int_0^{\delta t} z_l \, dz^j = \tfrac{1}{2}\left(z_l \cdot z^j - \delta_l^j\right) \cdot \delta t$$

Covariant Adjustments

- It is a useful simplification to fold in the factor convexities at this point

$$\tilde{\Gamma}_i^k \equiv \Gamma_i{}^k + \Delta_i \cdot \frac{L^i{}_j{}^{,k}}{L^i{}_j}$$

$$\tilde{\Theta} \equiv \Theta - \tfrac{1}{2} \cdot \left(\tilde{\Gamma}_i{}^k - \Gamma_i{}^k \right) \cdot \Sigma_k^i$$

Approximate Result

- The portfolio profit/loss in the finite case under the linear approximation is of the form below

$$\delta\Pi \approx \delta\Pi' \equiv T \cdot \delta t + \left(D_j \cdot z^j\right) \cdot \sqrt{\delta t} + \tfrac{1}{2} \cdot \left(z_l \cdot G^l{}_j \cdot z^j\right) \cdot \delta t$$

Coefficients

- Combine results from above to get the following

$$T \equiv \tilde{\Theta}\Big|_0 + \Delta_i\Big|_0 \cdot M^i\Big|_0$$

$$D_j \equiv \Delta_i\Big|_0 \cdot L^i{}_j\Big|_0$$

$$G^l{}_j \equiv L_k{}^l\Big|_0 \cdot \tilde{\Gamma}_i{}^k\Big|_0 \cdot L^i{}_j\Big|_0$$

Quadratic Form

- Since this is part of a quadratic form,
  - Symmetrize
  - Diagonalize
    - Eigenvalues are real; Transformation is orthogonal

$$G_l^{\,j} \equiv \tfrac{1}{2} \cdot \left(G^l{}_j + G_j{}^l \right)$$
$$\equiv O^j{}_k \cdot \tilde{G}_i^k \cdot O_l{}^i$$

Quadratic Form

- Re-express along Eigenvectors
  - Still independent standard Normal variants

$$\tilde{z}^i \equiv O_k{}^i \cdot z^k \quad , \quad \tilde{z}^i \sim \mathrm{iid}\,\mathrm{N}(0,1)$$

$$\tilde{D}_i \equiv D_j \cdot O^j{}_i$$

Approximate Result

- The result is a constant plus a sum of independent non-central chi-squared random variables

$$\delta\Pi' = T \cdot \delta t + \left(\tilde{D}_i \cdot \tilde{z}^i\right) \cdot \sqrt{\delta t} + \tfrac{1}{2} \cdot \left(\tilde{G}_i^i \cdot \tilde{z}_i \cdot \tilde{z}^i\right) \cdot \delta t$$

$$= \left(T - \frac{\tilde{D}_i^{\,2}}{2 \cdot \tilde{G}_i^i \cdot \delta t}\right) \cdot \delta t + \tfrac{1}{2} \cdot \tilde{G}_i^i \cdot \left(\tilde{z}^i + \frac{\tilde{D}_i}{\tilde{G}_i^i \cdot \sqrt{\delta t}}\right)^2 \cdot \delta t$$

Value-at-Risk

- The standard measure for market risk
  - Depends on time horizon & confidence level

$$\Pr\left\{\delta\Pi\left(\delta t\right) < -P\right\} \equiv \alpha$$

Value-at-Risk

- Standard parameters (Basle Accord)
  - Two-week time horizon
  - Three normal standard-deviation confidence level

$$\text{time horizon (yrs)} \quad \delta t = 14/365.25 = 0.0383\ldots$$

$$\text{confidence level} \quad \alpha = \Phi(-3) = 0.0013\ldots$$

Value-at-Risk

- Different approaches to solution
  - Central Limit Theorem
  - Monte Carlo simulation
  - Quadrature over conic sections
  - “Linearization”
    - Consistent with simpler approaches
    - Accounts for both sign & magnitude of gamma

Linearization

- Standardize each random variable

$$Z^i(\varepsilon) \equiv \operatorname{sgn}(\varepsilon) \cdot \frac{\left(\operatorname{sgn}\left(\tilde{D}_i\right) \cdot \tilde{z}^i + \varepsilon\right)^2 - \left(1 + \varepsilon^2\right)}{\sqrt{2 + 4 \cdot \varepsilon^2}}$$

Linearization

- Define the reciprocal of the non-centrality parameter as the curvature

$$K^i \equiv \frac{\tilde{G}_i^i}{\left|\tilde{D}_i\right|} \cdot \sqrt{\delta t}$$

$$\delta\Pi' = \left(T + \tfrac{1}{2} \cdot \tilde{G}_i^i\right) \cdot \delta t + \sqrt{1 + \tfrac{1}{2} \cdot K_i^{\ 2}} \cdot \left|\tilde{D}_i\right| \cdot Z^i \cdot \sqrt{\delta t}$$

Tailoring

- Scale the standardized variables by the ratio of the tail mass to the normal tail mass

$$\hat{Z}^i \equiv \frac{\Phi^{-1}(\alpha)}{F_{Z^i}^{-1}(\alpha)} \cdot Z^i$$

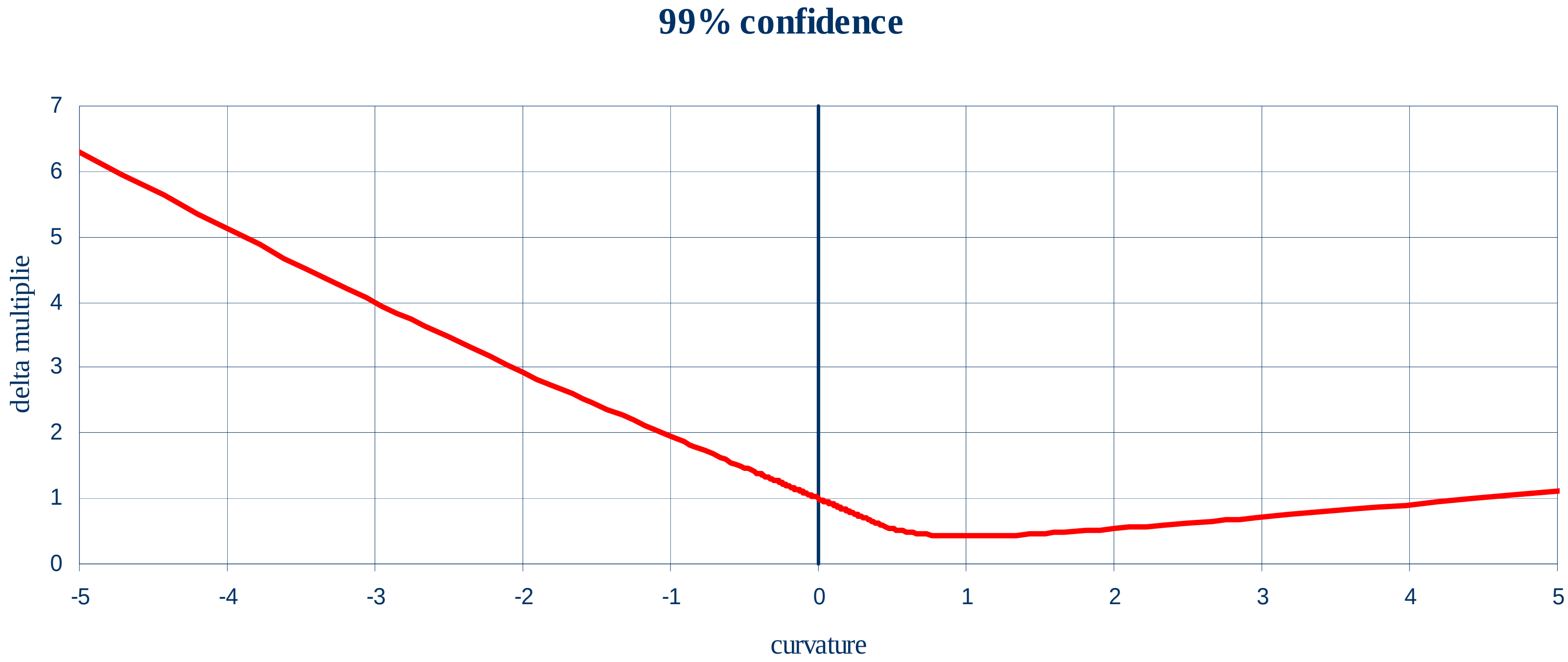
Tailoring

- Or equivalently, scale the delta
- And make one final adjustment to the theta

$$\hat{D}_i \equiv \left\{ \frac{F_{Z^i}^{-1}(\alpha)}{\Phi^{-1}(\alpha)} \cdot \sqrt{1 + \tfrac{1}{2} \cdot K_i^{\,2}} \right\} \cdot \tilde{D}_i$$

$$\hat{T} \equiv T + \tfrac{1}{2} \cdot \tilde{G}_i^i$$

Tailoring



99% confidence
delta multiplie
7
6
5
4
3
2
1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
curvature

Approximate Result

- All of this finally allows us to write the portfolio innovation in a quasi-linear form

$$\delta\Pi' = \hat{T} \cdot \delta t + \left|\hat{D}_i\right| \cdot \hat{Z}^i \cdot \sqrt{\delta t}$$

Normal Approximation

- Now, we apply a second approximation by replacing the standardized non-central chi-squared random variables by normal random variables

$$\delta\Pi' \approx \delta\Pi'' \equiv \hat{T} \cdot \delta t + \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t} \cdot z$$

$$\text{with} \quad z \sim \mathrm{N}(0,1)$$

"Linearized" Value-at-Risk

- It is straight-forward to solve for the Value-at-Risk in this case
- The drift may be suppressed as a third approximation

$$P'' = -\Phi^{-1}(\alpha) \cdot \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t} - \hat{T} \cdot \delta t$$

$$P'' \approx P''' \equiv -\Phi^{-1}(\alpha) \cdot \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t}$$