Risk & Asset Allocation
High Yield Bond Discussion

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Let's consider a stylized model for a defaultable bond and explore its properties.

Risk

For now, let's consider the valuation of this bond at the present moment, which we'll denote as $t = 0$ (time measured in years). The contract states that the bond pays a fixed coupon at rate $k > 0$ per year, which we will assume for simplicity to be paid continuously, until the maturity date $T > 0$ at which point it pays back the principal amount, which we will take to be 1. The value (price?) of the bond per unit of notional principal is P.

The bond's issuer is not currently in default, but it may default in the future. Let τ denote the random stopping time corresponding to default. If $\tau < T$, let's say that issuer immediately pays out some random fraction of the principal Y and ceases paying coupons[1].

The arbitrage-free value of the bond is the expected value of the risk-free discounted future payments under a risk-neutral probability measure. Let's take the (continuous) risk-free discounting rate to be r (constant) for simplicity.

$$P = \mathrm{E}\left[Ye^{-r(\tau \wedge T)} + (1-Y)e^{-rT}\chi_{\{\tau > T\}} + k \int_0^{\tau \wedge T} e^{-rt}\, dt \right] \tag{1}$$

where the random variables Y and τ are measured by $\mathbb{Q}$ conditional on the filtration $\mathcal{F}_0$. Notationally, $\tau \wedge T = \min(\tau, T)$ and $\chi_{\{\cdot\}}$ is the indicator function.

Let's suppose that τ and Y are independent, and that

$$\mathrm{E}[Y] = R$$
$$\mathrm{E}\left[\chi_{\{\tau > t\}}\right] = e^{-ht}$$

with parameters R representing the expected recovery and h the (constant) default intensity.

Valuation

Since

$$\mathrm{E}\left[\chi_{\{\tau \wedge T > t\}}\right] = e^{-ht}\chi_{\{t < T\}}$$

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[1] We will assume that the event $\{\tau = T\}$ is measure zero so we can ignore it.

the distribution and density function of $\tau \wedge T$ are

$$F_{\tau \wedge T}(t) = 1 - e^{-ht}\chi_{\{t<T\}}$$
$$f_{\tau \wedge T}(t) = he^{-ht}\chi_{\{t<T\}} + e^{-hT}\delta(t-T)$$

for $t > 0$. So we can integrate to get

$$\mathrm{E}\left[e^{-r(\tau \wedge T)}\right] = \frac{r}{r+h}e^{-(r+h)T} + \frac{h}{r+h}$$

We can put these together to evaluate (1):

$$P = \left(1 - \frac{k+Rh}{r+h}\right)e^{-(r+h)T} + \frac{k+Rh}{r+h} \tag{2}$$

Yield

There is a convention in the fixed-income markets to analyze bonds in terms of their (constant) internal rate of return, which is termed the bond's "yield". We will denote this by y. The yield on a bond depends on an "active redemption" assumption, which is typically taken to be the most conservative[2] fixed cashflow stream consistent with the bond's contract. In particular, one ignores the possibility of default in calculating the yield.

$$P \triangleq e^{-yT} + k\int_0^T e^{-yt}\,dt = \left(1 - \frac{k}{y}\right)e^{-yT} + \frac{k}{y}$$

Do not think of this as a valuation in the sense of (2). The expression above is merely a means of translating the bond's price into an alternate metric.

Identifying with our valuation (2), we get the correspondence

$$\left(1 - \frac{k+Rh}{r+h}\right)e^{-(r+h)T} + \frac{k+Rh}{r+h} = \left(1 - \frac{k}{y}\right)e^{-yT} + \frac{k}{y} \tag{3}$$

Let's review this result. There are two contractual variables, k and T; two pricing variables, y and r; and two risk management variables, h and R. Generally we can observe the first four and are interested in implying the the last two. With a single bond, that is generally not possible. With several bonds from the same issuer, it might be possible if we can assume they have the same expected recovery.

The case of a "par" bond is instructive:

$$P = 1 \quad \Longleftrightarrow \quad y = k = r + h(1-R)$$

That is, the yield spread on a par bond is the product of the default intensity and the complement of the expected recovery, which is also termed the "default severity".

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[2]conservative in the sense of lowest internal rate of return

Exposure

We are rarely in need of a valuation of a bond trading in the secondary market, because we can identify the bond's value with its market price. Investment mangers would typically be more interested in a results like (1) or (2) for risk management and sensitivity analysis. In this regard, the following result is notable:

$$\frac{\partial P}{\partial R} = \left(1 - \frac{r}{r+h}\right)\left(1 - e^{-(r+h)T}\right) \tag{4}$$

The bond price sensitivity to expected recovery is strictly increasing in default intensity and ranges from zero to one. For h small, the bond price is essentially insensitive to R. For h large, the bond price changes one-for-one with the expected recovery.

We can visualize this in Figure 1. We see that the price sensitivity to expected recovery rate is much greater for the high-yield bond (in red, note the vertical spread between the lines), then for the high-grade bond (in blue). In contrast we see that the price sensitivity to the default intensity, is somewhat greater for high-grade bond then the high-yield bond, except possibly for very low expected recoveries.

The analysis of these two dimensions of risk is very different, particularly in a portfolio setting. Intensity entails systemic concepts such as contagion, while expected recovery entails legal concepts such as collateral and seniority. Consequently, it is not surprising that fixed income investment managers tend to specialize is high-grade or high-yield bonds.
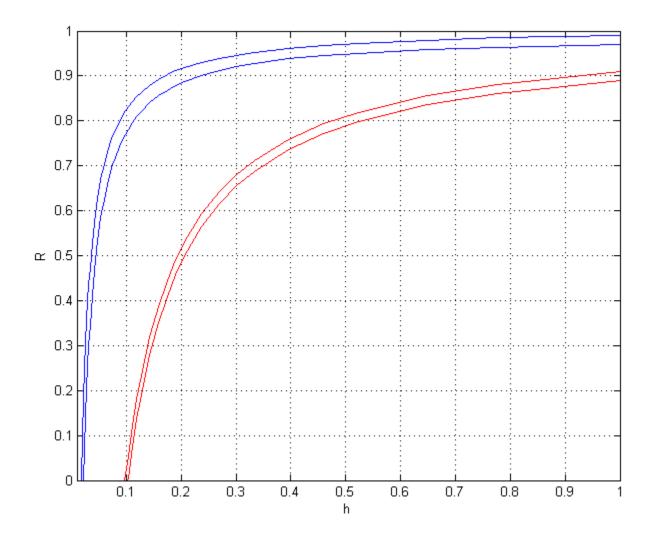
Figure 1: Level sets in (h, R) for $P \in \{0.99, 1.01\}$ and $k \in \{0.02, 0.10\}$ with $T = 5$ and $r = 0$.