Ch. 7 Intensity Models

Schönbucher, Philipp J. (2003)
Credit derivatives pricing models, Wiley Finance

John Dodson
finmath practitioner seminar

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Introduction

- chapter 7 is a continuation of chapter 3
- but we will drop Assumption 3.2, independence
- ...and specify (risk-neutral) stochastic dynamics
- since the building block securities can be expressed as

$$B(t,T) = \mathrm{E}^Q\left[e^{-\int_t^T r(s)\,ds} \middle| \mathcal{F}_t \right]$$
$$\bar{B}(t,T) = \mathrm{E}^Q\left[e^{-\int_t^T r(s)+\lambda(s)\,ds} \middle| \mathcal{F}_t \right]$$
$$e(t,T) = \mathrm{E}^Q\left[\lambda(T)\cdot e^{-\int_t^T r(s)+\lambda(s)\,ds} \middle| \mathcal{F}_t \right]$$

  we will draw inspiration from interest rate models

- N.B.: we assume $\exists\lambda$, a (risk-neutral) default intensity

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Introduction

- there are two main classes of interest rate models
  - short rate
  - forward rate
- short rate models can be tractable
- or at least amenable to efficient numerical techniques
  - tree methods
  - finite difference methods
- forward rate models have more flexible dynamics
- but are generally difficult to compute
  - simulation methods

Short rate models

short rate models start with a process description of the short-term interest rate evolving under the risk-neutral measure

- tractable models trade flexibility for simplicity & intuition
- each has certain flaws
  - Gaussian: potentially negative probabilities
  - Affine: strictly positive rate-spread correlation
- ...but they can be considered to be locally valid
- numerical models are much more flexible
- short-rate models are inherently Markovian, so efficient to evaluate

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Gaussian model

simplest stochastic intensity model

- the Gaussian model in §7.1 is based on the Vasicek model for interest rates
- the key dynamical assumption is that innovations in rates are normal
- Lemma 7.1 is the main analytical workhorse

$$dx(t) = (\kappa(t) - \alpha \cdot x(t))\ dt + \sigma(t)\ dW(t)$$
$$\implies \mathrm{E}\left[e^{-\int_t^T x(s)\ ds} \middle| \mathcal{F}_t \right] = e^{\mathcal{A}(t,T) - \mathcal{B}(t,T)\cdot x(t)}$$

  where $\mathcal{A}$ and $\mathcal{B}$ are defined in (7.10) and (7.9)
- Proposition 7.2 uses these to derive solutions for the building blocks where $r(t)$ and $\lambda(t)$ are combinations of two Gaussian processes

Multifactor gaussian model

general framework for correlation

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- the factors are latent in the multifactor gaussian model
- instead we start with the risk-neutral process definitions of the zerobond values

$$\frac{dB(t,T)}{B(t,T)} = r(t)\ dt + \vec{a}(t,T)'\ d\vec{W}(t)$$
$$\frac{d\bar{B}(t,T)}{\bar{B}(t-,T)} = (r(t) + \lambda(t))\ dt + \vec{\bar{a}}(t,T)'\ d\vec{W}(t) - dN(t)$$

  where the vector form of the volatility allows for a completely general description of the correlation of innovations between the curves and amongst points on the curves over time

Multifactor gaussian model

general framework for correlation

- in this setting, where B and $\bar{B}$ are taken as given, the remaining building block security is

$$e(t,T) = \bar{B}(t,T)\cdot \left[h(t,T) - \int_t^T \vec{\bar{a}}(s,T)' \frac{\partial}{\partial T}\left(\vec{\bar{a}}(s,T) - \vec{a}(s,T)\right)\, ds\right]$$

  where $h = \bar{f} - f$ is the forward hazard rate
- the last term reflects the correlation between r and λ
- N.B.: I believe there are typos in (7.23) and (7.24)

Cox-Ingersoll-Ross model

- the affine model in §7.2-3 is based on the Cox-Ingersoll-Ross model for interest rates
- the key dynamical assumption is that innovations in rates are ANC: affine combinations of non-central chi-squared random variables
- say there are n independent factors x_i, where

$$dx_i(t) = (\alpha_i - \beta_i \cdot x_i(t))\; dt + \sigma_i \cdot \sqrt{x_i(t)}\, dW_i(t)$$

  with positive parameters and $\alpha_i > \sigma_i^2/2$ to insure that $x_i(t) > 0$ a.s.
- analogously to the gaussian model, the main analytical workhorse for this model is

$$\mathrm{E}\left[e^{-\int_t^T \sum_i c_i \cdot x_i(s)\, ds} \middle| \mathcal{F}_t \right] = e^{\sum_i \log H_{1i}(T-t,c_i) - H_{2i}(T-t,c_i)\cdot c_i \cdot x_i(t)}$$

  where H_1 and H_2 are defined in (7.29) and (7.30)

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Cox-Ingersoll-Ross model

- define the short rate and default intensity by

$$r(t) = \sum_{i=1}^{n} w_i \cdot x_i(t)$$
$$\lambda(t) = \sum_{i=1}^{n} \bar{w}_i \cdot x_i(t)$$

  with w_i and $\bar{w}_i$ non-negative
- this guarantees that rates are non-negative
- ...but also limits the model to positive correlations
- there are a total of $5n+2$ parameters to fit to calibrate the model

Cox-Ingersoll-Ross model

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with the workhorse and the result of Proposition 7.8, we can write down the values of the building block securities

$$B(t,T) = e^{\sum_{i=1}^{n} \log H_{1i}(T-t,w_i) - H_{2i}(T-t,w_i)\cdot w_i \cdot x_i(t)}$$
$$\bar{B}(t,T) = e^{\sum_{i=1}^{n} \log H_{1i}(T-t,w_i+\bar{w}_i) - H_{2i}(T-t,w_i+\bar{w}_i)\cdot (w_i+\bar{w}_i) \cdot x_i(t)}$$

and

$$e(t,T) = \bar{B}(t,T)\cdot$$
$$\sum_{i=1}^{N} \bar{w}_i \cdot (w_i + \bar{w}_i) \cdot \left(\alpha_i + x_i(t) \cdot \frac{\partial}{\partial T}\right) H_{2i}(T-t, w_i + \bar{w}_i)$$

Tree model

- the tree model in §7.4 is based on the Hull-White trinomial model for interest rates
  - which in turn is based on the gaussian model
- there are a total of ten successors to each node
  - although most of these are successors are shared—the tree is re-combining
- in order to prevent rates from going negative, the excursion is artificially limited by having the tree fold back on itself
- the risk-neutral branching probabilities are calibrated according to
  - (7.78-86) to fit the moments of the dynamics
  - tables 7.1-3 to incorporate the correlation
- note that tree models are essentially explicit schemes that implement the PDE in the next section

Tree model

Hull-White gaussian model

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in the Hull-White short rate model, the mean reversion level is allowed to vary deterministically in order to facilitate calibration

$$dr(t) = [k(t) - a \cdot r(t)]\ dt + \sigma\ dW(t)$$

the solution to this is

$$r(t) = r^*(t) + \alpha(t)$$

where the auxiliary process, r^*, is defined by $r^*(0) = 0$ and

$$dr^* = -a \cdot r^*\ dt + \sigma\ dW$$

and the deterministic offset

$$\alpha(t) = r(0) \cdot e^{-a \cdot t} + \int_0^t e^{-a \cdot (t-s)} \cdot k(s)\ ds$$

Tree model

Hull-White trinomial tree model

- the model has six parameters: a, σ, $\bar{a}$, $\bar{\sigma}$, ρ, and π
- once these are specified,
  - trees in r^* and λ^* can be constructed; and
  - $\alpha(\{0:T\})$ and $\bar{\alpha}(\{0:T\})$ can be fit to $B(0,\{0:T\})$ and $\bar{B}(0,\{0:T\})$ by forward induction, starting with the risk-free curve
- the risk-neutral default branching probability is $p = 1 - e^{-\lambda\cdot\Delta t}$ from each non-default node
  - the recovery in the default node is π
  - should it be needed, the local expectation of the default time is $\tau^e = t + \frac{1}{\lambda}\cdot\left(1 - \frac{\lambda\cdot\Delta t}{e^{\lambda\cdot\Delta t}-1}\right)$
- once calibrated, the tree can be used to price defaultable securities by backwards induction
- early exercise can be modeled by evaluating the early exercise option at each step

Partial differential equation

- to derive the partial differential equation for a defaultable security in the intensity setting, start by specifying the stochastic processes for the rates

$$dr = \mu_r\,dt + \sigma_r\,dW_1$$
$$d\lambda = \mu_\lambda\,dt + \sigma_\lambda\cdot\left(\rho\,dW_1 + \sqrt{1-\rho^2}\,dW_2\right)$$

- also, let the compensator measure for the marked point process be

$$\nu(dt, d\pi) = \lambda(t)dt\;K(d\pi)$$

- and let the cashflow densities be

$\tilde{f}(t,r,\lambda)$	prior to default
$g(t,r,\lambda,\pi)$	at default

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Partial differential equation

- define the value of the security to be

$$V(t) = v\,(t, r(t), \lambda(t)) \quad \text{for} \quad t < \tau$$

- applying Itô's Lemma and the fundamental pricing rule, we get that v must satisfy

$$\partial_t v + \mathcal{L}v - (r + \lambda) \cdot v = -g^e \cdot \lambda - \tilde{f}$$

where the diffusion's linear operator is

$$\mathcal{L} = \mu_r \cdot \partial_r + \tfrac{1}{2} \cdot \sigma_r^2 \cdot \partial_{rr} + \mu_\lambda \cdot \partial_\lambda + \tfrac{1}{2} \cdot \sigma_\lambda^2 \cdot \partial_{\lambda\lambda} + \rho \cdot \sigma_r \cdot \sigma_\lambda \cdot \partial_{\lambda r}$$

and the locally (risk-neutral) expected default payoff is

$$g^e(t, r, \lambda) = \int_0^1 g(t, r, \lambda, \pi) K(d\pi)$$

Partial differential equation

In these terms, a valuation formula v for any defaultable claim can be found given:

1. a final condition defined in terms of the no-default payoff *(may not be so simple)*

$$v(T, r, \lambda) = F(r, \lambda)$$

2. boundary conditions on v at the extremes of r and λ

- the problem is comparable to that for a default-free interest-sensitive claim
- in particular, adding a stochastic intensity and recovery effectively transforms:

$$\begin{aligned} &\text{discount rate} \quad r \to r + \lambda \\ &\text{dividend rate} \quad \tilde{f} \to \tilde{f} + g^e \end{aligned}$$

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Forward rates framework

- the whole curve model in §7.6 is based on Heath-Jarrow-Morton
  - all short rate models are consistent with HJM
  - HJM with deterministic volatility is equivalent to the gaussian model
- since the dynamics are essentially unlimited, need to specify the no-arbitrage drift restriction
- model is non-Markovian, so valuation is in a simulation setting
- an implementation would proceed in the following steps
  1. specify initial risk-free term structure $f(0, \{0 : T\})$
  2. ...zero-recovery spread term structure $h(0, \{0 : T\})$
  3. ...forward rate volatilities and correlations
  4. calculate drifts
  5. simulate paths in β, B, $\bar{B}$, e, I, and π
  6. evaluate pathwise security values
  7. average

Heath-Jarrow-Morton

we start by taking as given,

1. the rates and spreads for $T \geq 0$

$$f(0, T) \quad \text{and} \quad h(0, T)$$

2. and the $i = 1, \ldots, n$ co-volatilities for $t \geq 0$ and $T \geq t$

$$\sigma_i(t, T) \quad \text{and} \quad \sigma_i^h(t, T) \quad \forall t < \tau$$

- and define the risk-neutral processes for $0 \leq t \leq T$,

$$df(t, T) = \alpha(t, T)\, dt + \sum_{i=1}^{n} \sigma_i(t, T)\, dW_i$$

$$dh(t, T) = \alpha^h(t, T)\, dt + \sum_{i=1}^{n} \sigma_i^h(t, T)\, dW_i \quad \forall t < \tau$$

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Heath-Jarrow-Morton

no-arbitrage imposes the drift restrictions

$$\alpha(t,T) = \sum_{i=1}^{n} \sigma_i(t,T) \cdot \int_t^T \sigma_i(t,T')\, dT'$$

and

$$\alpha^h(t,T) = \sum_{i=1}^{n} \Big[\sigma_i^h(t,T) \cdot \int_t^T \sigma_i(t,T')\, dT' + \Big(\sigma_i(t,T) + \sigma_i^h(t,T)\Big) \cdot \int_t^T \sigma_i^h(t,T')\, dT' \Big]$$

also, Proposition 7.11 verifies that

$$h(t,t) = \lambda(t)$$

which can be used to model the default indicator process

Monte Carlo

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to value a defaultable security under an intensity model, we need to evaluate the pathwise integral in (7.143),

$$\mathrm{E}^Q \left[\beta(0,T) \cdot I(T) \cdot F(T) \middle| \mathcal{F}_0 \right] + \mathrm{E}^Q \left[\beta(0,\tau) \cdot N(T) \cdot g(\tau,\pi) \middle| \mathcal{F}_0 \right] \mathrm{E}^Q \left[\int_0^T \beta(0,t) \cdot I(t) \cdot \tilde{f}(t)\, dt \middle| \mathcal{F}_0 \right]$$

where

- F is the final payoff at T if $T < \tau$
  - may depend on $f(T,t)$ and $h(T,t)$ for $T \le t < \bar{T}$
- $g(t,\pi)$ is the payoff at default for recovery π if $t = \tau$
- $\tilde{f}(t)$ is the dividend rate for $t < \tau \wedge T$

and β, τ, N, I, and π all depend on the path ω

Monte Carlo

scheme for rates

the HJM framework is well-suited for an Euler scheme to simulate pathwise forward rates

- time from $t = 0$ to $\bar{T} \geq T$ is discretized
- forward rates are updated according to

$$f(t_m, t_l) = f(t_{m-1}, t_l) + \alpha(t_m, t_l) \cdot (t_m - t_{m-1}) + \sum_{i=1}^{n} \sigma_i(t_m, t_l) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}}$$

$$h(t_m, t_l) = h(t_{m-1}, t_l) + \alpha^h(t_m, t_l) \cdot (t_m - t_{m-1}) + \sum_{i=1}^{n} \sigma_i^h(t_m, t_l) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}}$$

for $l \geq m$ and $\epsilon_{i,m}$ standard i.i.d. variates

Monte Carlo

schemes for default

by way of illustration, the book describes three schemes of increasing efficiency for simulating default

- fixed time grid
  - draw a uniform variate at each step in each path
  - default immediately if $-\log U_m < h(t_m, t_m) \cdot (t_m - t_{m-1})$
- direct simulation of default time
  - draw a single uniform variate for each path
  - set τ to be the lowest t_M with $-\log U < \sum_{m=1}^{M} h(t_m, t_m) \cdot (t_m - t_{m-1})$
- simulation with branching to default
  - do not simulate default at all; apply iterated expectations instead

the author claims the latter method converges much quicker

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simulation with branching to default

analogously to Figure 3.2, calculate the (risk-neutral) expected defaultable value at each step in the tree

for each path,

- simulate all rates f and h out to $\bar{T}$
- set $\beta = \gamma = 1$ and $v = 0$ at $t = 0$
- value the security along the path from $t = 0$ to T
  1. update the discount factor: $\beta \leftarrow \beta \cdot e^{-f(t,t) \cdot \Delta t}$
  2. calculate the default probability: $p = 1 - e^{-h(t,t) \cdot \Delta t}$
  3. update the survival probability: $\gamma \leftarrow \gamma \cdot (1 - p)$
  4. accumulate the value:
     $v \leftarrow v + \beta \cdot \gamma \cdot \left(\tilde{f}(t) \cdot \Delta t \cdot (1 - p) + g^e(t) \cdot p \right)$
     - where $g^e(t) = \mathrm{E}^Q \left[g(t, \pi) \mid \mathcal{F}_t \right]$
     - N.B.: include the payoff $F(T)$ in the last step

...then average the values, v, over the paths

Conclusion

- questions?
- remaining chapters

ch. 5	Cox process	Chris Bemis
ch. 6	*recovery models*	
ch. 8	transition models	John Baxter
ch. 9	structural model	Bill Barr
ch. 10	correlation models	Carlos Tolmasky

- papers
  - Duffie-Lando (2001), Term structures of credit spreads with incomplete accounting information
  - Andersen-Sidenius-Basu (2003), All your hedges in one basket
  - Carr-Flesaker (2006), Robust replication of default contingent claims
  - Errais-Giesecke-Goldberg (2007), Pricing credit from the top down with affine point processes

Calendar

seven Wednesdays left before Fall term starts on September 4 (Labour Day is September 3)

- July 18
- July 25
- August 1
- August 8
- August 15
- August 22
- August 29 (State Fair!)