Exact Results for The Structural Model with Perpetual Debt

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Abstract

The structural model for a firm with a single aggregate source of uncertainty is especially tractable in a perpetual setting. I show closed-form results for the densities of the horizon value of the firm's aggregate equity and debt, and arbitrage-free values of various securities and derivatives, including European-style equity options and credit default swaps.

Keywords: structural model; perpetual debt; endogenous default; credit default swap; European option; leverage effect.

1 Introduction

The structural model of the firm was introduced by Black & Scholes (1973). In the example they laid out, the firm's liabilities consisted of a single discount bond. Upon maturity, the firm would be either wound-up or recapitalized. There were no cashflows at all.

This was ideal for the application of the new formulae for European-style options that were introduced in that paper; and in spite of the fact that the fixed-term setting directly contradicts the fundamental principle of the firm as a going concern, the precedent has held. To this day, the structural model is presented in introductory graduate finance texts in terms of a pure discount liability structure. See, for example, the treatment in McDonald (2006).

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An alternate perspective appeared shortly thereafter, when Merton's (1974) paper on corporate debt was published. In the last section, the author re-opened the discussion of coupons, and even perpetual debt; but the absence of a dividend policy and an awkward solution involving the confluent hypergeometric function seems to have hindered its subsequent development.[1]

The application of the classical Black-Scholes-Merton analysis to debt reached its apex shortly thereafter in Black & Cox (1976). This paper introduced models incorporating discrete and continuous coupons and American-style defaults specified exogenously or derived endogenously. They presented the tractability of continuous perpetual debt, which has served as the starting point for much subsequent work, including this paper.

The subject would have to wait almost twenty years for significant analytical advancement. The Longstaff & Schwartz (1995) extension introduced stochastic interest rates. Here, default is strictly exogenous. Furthermore, reliance upon the affine class precluded its widespread application in industry. This model was generalized in Collin-Dufresne & Goldstein (2001) to accommodate several empirical findings.

In the same period, Leland (1994) repeated Merton's analysis of perpetual debt while avoiding payouts altogether. This was extended in Leland & Toft (1996) to handle discrete principal payments. Their motivation was the determination of optimal capital structure; and while they obtained results similar to mine, they did not (and could not) apply the framework to the valuation of securities.

I submit that a thoughtful treatment of liability and equity cashflows results in a version of the structural model with endogenous default that is not only economically plausible; but is analytically simpler and more powerful then its predecessors.

2 Balance Sheet Model

Let us model the micro-economics of a firm with publicly-traded debt and equity as a continuous and perpetual stream of payments to investors funded by the economic value added of the firm.

Let us assume a single class of debt and model the debtholders' interest in the underlying assets of the firm, S_t, in terms of a perpetual American-style put struck at some level, K, representing the indebtedness of the firm[2] producing a perpetual defaultable interest stream to debtholders at fixed rate $r \cdot K$ where r is risk-free interest. Let us also assume that the assets of the firm produce a perpetual cashflow stream at variable rate $\delta \cdot S_t$ which is used to service the debt and pay dividends to owners. I do not explicitly model wages, taxes, or any other costs of production. We will assume that the value of the underlying

[1] This solution has nonetheless persisted. See, for example, the binomial-tree version of the structural model in Ho & Lee (2004).

[2] Note that K is *not* the face value of the debt, nor is it the default threshold.

economic assets of the firm, S_t, follows a geometric brownian motion with *ex div* drift $\mu - \delta$ and volatility σ.

2.1 Equity and Debt

Let us start with the Merton (1973) solution for the perpetual put value in this setting,

$$p_t = (K - S_t \wedge L) \cdot \left(\frac{L}{S_t \vee L} \right)^{\gamma} \tag{2.1}$$

with

$$\gamma = \frac{r-\delta}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2 \cdot r}{\sigma^2}} \tag{2.2}$$

$$L = \frac{K}{1 + \frac{1}{\gamma}} \tag{2.3}$$

which is verified in Appendix B. Note that γ is a positive[3] dimensionless quantity and $L < K$.

The put value rises as S_t falls. For $S_t \leq L$ the value becomes the intrinsic—indicating that it is optimal for the owners to exercise—the first time S_t falls through L. This represents liquidation.

Prior to liquidation the aggregate debt is worth

$$D_t = K - p_t \tag{2.4}$$

Modeling this as a perpetual annuity whose value is $K \cdot r/(r + s_t)$, we can see that the implied perpetual spread is

$$s_t = r \cdot \frac{p_t}{D_t} \tag{2.5}$$

This increases to a maximum value of r/γ as S_t falls to L. I will discuss the yield spread distribution in Section 5.2 and the term structure in Section 6.1.

At liquidation the debtholders' claim is worth L, representing the aggregate recovery value.

Prior to liquidation the aggregate owners' equity is worth

$$E_t = S_t - K + p_t \tag{2.6}$$

2.2 Black-Scholes

Considering equity and debt as derivatives on the underlying assets, let us review the greeks.

[3] $\lim_{\sigma \to \infty} \gamma = 0$

The deltas of the equity and debt with respect to the underlying asset value are

$$0 < \frac{\partial E_t}{\partial S_t} = 1 - \gamma \cdot \frac{p_t}{S_t} < 1 \tag{2.7}$$

$$0 < \frac{\partial D_t}{\partial S_t} = \gamma \cdot \frac{p_t}{S_t} < 1 \tag{2.8}$$

the gammas are

$$\frac{\partial^2 E_t}{\partial S_t^2} = \gamma \cdot (\gamma + 1) \cdot \frac{p_t}{S_t^2} > 0 \tag{2.9}$$

$$\frac{\partial^2 D_t}{\partial S_t^2} = -\gamma \cdot (\gamma + 1) \cdot \frac{p_t}{S_t^2} < 0 \tag{2.10}$$

and the thetas are zero by design.

I verify in Appendix B that these formulations satisfy the Black-Scholes equation for no-arbitrage with cashflow rates $\delta \cdot S_t - r \cdot K$ for the equity and $r \cdot K$ for the debt.

3 Default Model

The enterprise value follows geometric brownian motion; in particular,

$$S_T(\omega) = S_0 \cdot e^{\left(\mu - \delta - \frac{1}{2} \cdot \sigma^2\right) \cdot T + \sigma \cdot B_T(\omega)} \tag{3.1}$$

where $B_t(\omega)$ is a realization of a standard brownian motion with respect to the standard probability triple and $\omega \in \Omega$ which we will henceforth suppress.

3.1 Liquidation

The probability that the firm is liquidated before time T is

$$\begin{aligned} q_T &= \Pr_0 \left\{ \min_{0<t<T} S_t < L \right\} \\ &= \Pr_0 \left\{ \min_{0<t<T} \left[B_t + \frac{\mu - \delta - \frac{1}{2} \cdot \sigma^2}{\sigma} \cdot t \right] < - \frac{\log S_0/L}{\sigma} \right\} \end{aligned}$$

Using results from Appendix D about the distribution of the maximum excursion of a brownian motion, we can demonstrate that

$$\begin{aligned} q_T = \left(\frac{S_0}{L} \right)^{1 - 2 \cdot \frac{\mu - \delta}{\sigma^2}} &\cdot \Phi \left(\frac{- \log S_0/L + \left(\mu - \delta - \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}} \right) \\ &+ \Phi \left(\frac{- \log S_0/L - \left(\mu - \delta - \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}} \right) \end{aligned} \tag{3.2}$$

where $\Phi(\cdot)$ is the standard normal CDF[4].

In the limit $T \to \infty$,

$$q_\infty = \begin{cases} \left(\frac{S_0}{L}\right)^{1-2\cdot\frac{\mu-\delta}{\sigma^2}} & \mu > \delta + \frac{1}{2}\cdot\sigma^2 \\ 1 & \text{otherwise} \end{cases}$$

This means that if the asset growth rate is too low, the firm is guaranteed to eventually go into liquidation; while if the asset growth rate is high enough, there is a chance (but not a guarantee) that the firm will avoid liquidation indefinitely.

We also know from the appendix that $q_0 = 0$, and generally

$$\lim_{T\to 0} \frac{q_T}{T^n} = 0 \quad n = 1, 2, \ldots$$

for $S_0 > L$, which in particular means that there can be no hazard rate associated with liquidation (or any other passage event) in this model. Technically, default is said to be an accessible event.

3.2 Financial Distress

From Appendix B, it would seem that the equity dividends are negative for $L < S_t < \frac{r}{\delta}\cdot K$. If the owners have limited liability, then this is prevented. Instead, interest payments due to debtholders are missed and the firm is compelled to enter bankruptcy protection. In this model, it is not until the value of the firm's assets falls to L that the debtholders can expect to take control of the assets.

Let us denote the bankruptcy boundary L'.

$$L' = \frac{r}{\delta}\cdot K = \frac{L}{1-\gamma\cdot\frac{\sigma^2}{2\cdot r}}$$

Since $L' > L$, the firm will always experience financial distress prior to liquidation. It is possible that the firm may recover if S_t subsequently wanders above L' before it hits the absorbing L. Let us assume that any unpaid interest accrued during a successful re-organization is ultimately paid to the creditors.

One might imagine a "fuzzy boundary" defining default somewhere in the asset value range between L' and L, since the bankruptcy process entails extra costs and additional uncertainty. I will no go into this further here.

4 Increments

4.1 Finite Increments

As a function of the brownian motion, the owners' equity value at time T is

$$E_T = L\cdot f_\gamma\left(f_\gamma^{-1}\left(\frac{E_0}{L}\right)\cdot e^{\left(\mu-\delta-\frac{1}{2}\cdot\sigma^2\right)\cdot T+\sigma\cdot B_T}\right) \tag{4.1}$$

[4] $\Phi(0) = \frac{1}{2}$ and $\Phi'(z) = e^{-z^2/2}/\sqrt{2\pi}$.

where

$$f_\gamma(x) = x - 1 - \frac{1 - x^{-\gamma}}{\gamma} \tag{4.2}$$

provided

$$\min_{0<t<T} \left[B_t + \frac{\mu - \delta - \frac{1}{2} \cdot \sigma^2}{\sigma} \cdot t \right] \geq - \frac{\log f_\gamma^{-1}\left(E_0/L\right)}{\sigma}$$

The horizon value of the debt is similar, with $x - f_\gamma(x)$ in place of $f_\gamma(x)$.

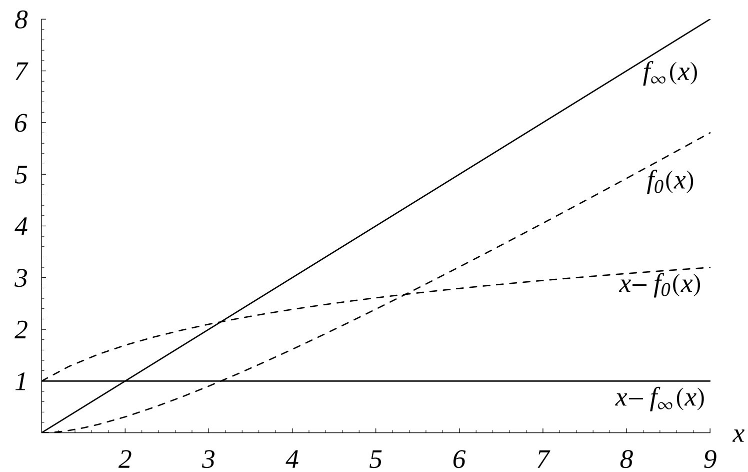

Figure 1: Normalized debt and equity for limiting values of γ.

4.2 Instantaneous Increments

Define the (simple) total return on the equity over the period $t \in (0, T]$ to be

$$R_T^E = \frac{E_T + \delta \cdot \int_0^T S_t \, dt - r \cdot K \cdot T - E_0}{E_0 \cdot T} \tag{4.3}$$

Applying Itô's Lemma to $R_T^E \cdot T$, we can see that

$$\mathsf{E}_0 \left\{ \lim_{T \to 0} R_T^E \right\} = r + (\mu - r) \cdot \Omega_0 \tag{4.4}$$

and

$$\sqrt{\mathsf{var}_0 \left\{ \lim_{T \to 0} R_T^E \cdot \sqrt{T} \right\}} = \sigma \cdot \Omega_0 \tag{4.5}$$

where Ω_t is the equity elasticity due to financial leverage,

$$\Omega_t = \frac{S_t}{E_t} \cdot \frac{\partial E_t}{\partial S_t} = \frac{S_t - L}{E_t} \cdot (1 + \gamma) - \gamma \tag{4.6}$$

These are consistent with the results from Appendix C.

Notice that the equity volatility is simply the asset volatility scaled by the elasticity, and the instantaneous risk-adjusted excess total return for the equity is that of the underlying assets, namely $(\mu - r)/\sigma$.

The total return on the debt is

$$R_T^D = \frac{D_T + r \cdot K \cdot T - D_0}{D_0 \cdot T} \tag{4.7}$$

which has drift rate

$$\mathsf{E}_0 \left\{ \lim_{T \to 0} R_T^D \right\} = r + (\mu - r) \cdot \left(1 - (\Omega_0 - 1) \cdot \frac{E_0}{D_0} \right) \tag{4.8}$$

and volatility

$$\sqrt{\mathsf{var}_0 \left\{ \lim_{T \to 0} R_T^D \cdot \sqrt{T} \right\}} = \sigma \cdot \left(1 - (\Omega_0 - 1) \cdot \frac{E_0}{D_0} \right) \tag{4.9}$$

The debt has the same instantaneous risk-adjusted excess total return as the assets, but generally much lower volatility[5].

Instantaneous debt and equity total returns are perfectly correlated, since there is only one source of uncertainty. In fact, equity and debt are substitutes for one other (and the underlying assets) from a mean-variance perspective in the instantaneous total return setting. Differences emerge in the finite setting because of the possibility of default. And even in the instantaneous setting, we see from Appendix C that the higher moments of debt, equity, and assets differ, with equity innovations more positively skewed than assets and debt innovations negatively skewed.

5 Densities

5.1 Equity Density

We can use the results from Appendix D to write down the density function of the horizon value of equity. In the notation of the appendix,

$$E_T = L \cdot f_\gamma \left(e^{\sigma \cdot \left(\tilde{B}'_T - M \right)} \right) \tag{5.1}$$

and

$$\theta = \frac{\mu - \delta - \frac{1}{2} \cdot \sigma^2}{\sigma} \tag{5.2}$$

$$M = -\frac{\log f_\gamma^{-1} \left(\frac{E_0}{L} \right)}{\sigma} \tag{5.3}$$

[5] For γ greater than one-half, the debt volatility does not exceed about one-sixth of the equity volatility (for constant risk-free interest rates).

where $f_\gamma^{-1} : [0,\infty) \mapsto [1,\infty)$ is well-defined for $\gamma > 0$ but unfortunately does not seem to have a more primitive expression.

Since

$$x \cdot f'_\gamma(x) = (x-1)\cdot(1+\gamma) - f_\gamma(x)\cdot\gamma \tag{5.4}$$

we can apply a change of variable to the result (D.15) to get

$$\begin{aligned}\mathsf{Pr}_0\{E_T \in dy\} = &\left(1 - e^{-2\cdot\log f_\gamma^{-1}(y/L)\cdot\log f_\gamma^{-1}(E_0/L)/\left(\sigma^2\cdot T\right)}\right)\\ &\cdot \frac{e^{-\frac{1}{2}\cdot\left(\log f_\gamma^{-1}(y/L) - \log f_\gamma^{-1}(E_0/L) - \left(\mu-\delta-\sigma^2/2\right)\cdot T\right)^2/\left(\sigma^2\cdot T\right)}}{\left(L\cdot\left(f_\gamma^{-1}(y/L)-1\right)\cdot(1+\gamma) - y\cdot\gamma\right)\cdot\sqrt{2\pi\cdot\sigma^2\cdot T}}\,dy\end{aligned} \tag{5.5}$$

for $y > 0$. We can see that the equity is approximately log-normal with volatility $\sigma\cdot\Omega_0$.

Keep in mind that the equity density has a pole at zero, not represented here, corresponding to liquidation whose magnitude is given by q_T from equation (3.2).

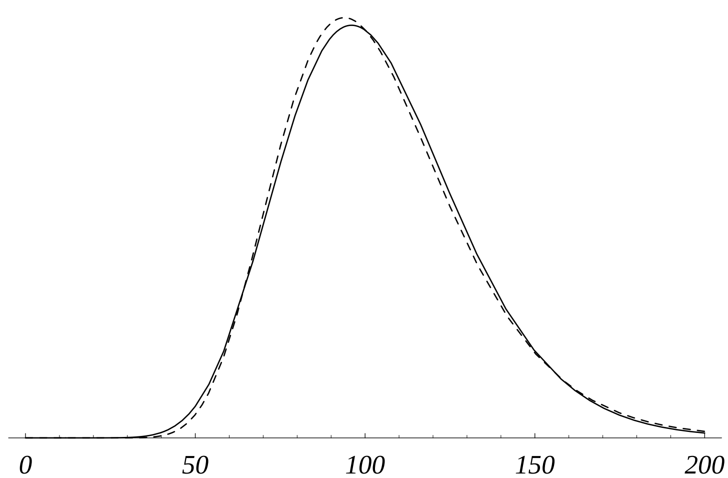

Figure 2: Equity density at $T = 1$ for $E_0 = 100$ with $r = 0.05$, $\mu = 0.06$, $\delta = 0.03$, $\sigma = 0.2$, and $L = 20$. Dashed line is the corresponding log-normal approximation.

5.2 Spread Density

Similarly, we can use the definition of the perpetual spread,

$$s_t = \frac{r}{(1+\gamma)\cdot\left(\frac{S_t}{L}\right)^\gamma - 1} \tag{5.6}$$

to change variables in (D.15) to get its horizon density,

$$\mathsf{Pr}_0\left\{s_T \in dy\right\} = \left(1 - e^{-2\cdot\log\frac{1+r/s_0}{1+\gamma}\cdot\log\frac{1+r/y}{1+\gamma}/\left(\gamma^2\cdot\sigma^2\cdot T\right)}\right)$$
$$\cdot\frac{e^{-\frac{1}{2}\cdot\left(\log\frac{1+r/y}{1+r/s_0}+\left(r-\gamma\cdot(\mu-r)-\gamma^2\cdot\sigma^2/2\right)\cdot T\right)^2/\left(\gamma^2\cdot\sigma^2\cdot T\right)}}{y^2/r\cdot(1+r/y)\cdot\sqrt{2\pi\cdot\gamma^2\cdot\sigma^2\cdot T}}\,dy \quad (5.7)$$

on the compact domain $(0, \frac{r}{\gamma})$. Note that the spread becomes undefined upon liquidation, so this is an improper density whose total mass is $1-q_T$. Notice that for s_t small, the quantity $1 + r/s_t$ is approximately log-normal with volatility $\gamma \cdot \sigma$.

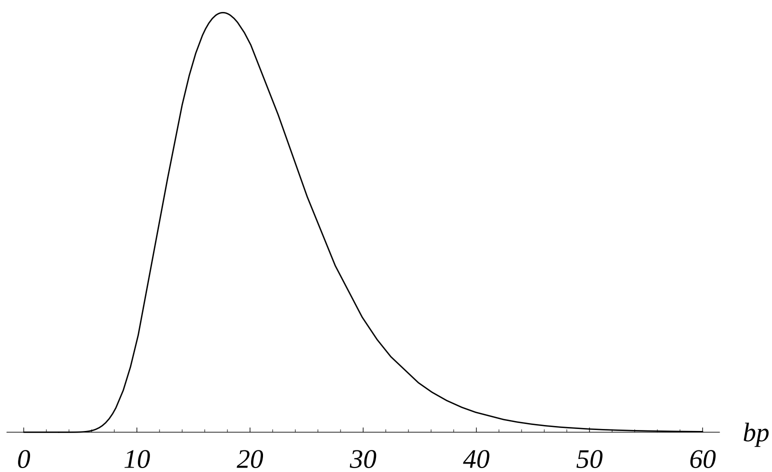

Figure 3: Spread density at $T = 1$ for $s_0 = 20$ bp with $r = 0.05$, $\mu = 0.06$, $\delta = 0.03$, $\sigma = 0.2$.

6 Derivatives

6.1 Credit Default Swaps

Rather than pricing individual debt obligations that make up the total debt capitalization $D_0 = D(S_0)$, let us consider the pricing of a credit default swap that gives the right to immediately recover the current value of an underlying nominal perpetual debenture in case of a future credit event.

Define the credit event by the passage $S_\tau = L'$ when the asset value falls to a level where earnings no longer support interest payments. Say a swap is originated on date t with maturity T. The swap payoff in the event of default at time $\tau \leq T$ is $1 - D(L')/D(S_t)$ per unit of notional. The discounted risk-neutral expected value of the payoff is funded by an annuity with maturity $\tau \wedge T$ whose constant payment is the T-year CDS-implied yield spread.

The spread is therefore defined by

$$0 = \tilde{\mathsf{E}}_t \left\{ s_{\text{swap}}(t,T) \cdot \frac{1 - e^{-r\cdot((\tau\wedge T)-t)}}{r} - \left(1 - \frac{D\left(L'\right)}{D\left(S_t\right)}\right) \cdot H(T-\tau) \cdot e^{-r\cdot(\tau-t)} \right\}$$

or

$$s_{\text{swap}}(t,T) = r \cdot \frac{1 - \frac{D\left(L'\right)}{D\left(S_t\right)}}{\frac{1-e^{-r\cdot(T-t)}\cdot\left(1-\tilde{\mathsf{P}}\mathsf{r}_t\{\tau<T\}\right)}{\tilde{\mathsf{E}}_t\left\{e^{-r\cdot(\tau-t)}\cdot H(T-\tau)\right\}} - 1} \tag{6.1}$$

in general for $t < T$.

We can use (D.20) to evaluate the risk-neutral expectations, obtaining the following result[6].

$$s_{\text{swap}}(t,T) = s_t \cdot \frac{d_t - 1}{d_t \cdot \frac{1-e^{-r\cdot(T-t)}\cdot\left(\Phi\left(\frac{\log d_t - \left(r-\sigma'^2/2\right)\cdot(T-t)}{\sigma'\cdot\sqrt{T-t}}\right) - d_t^{2\cdot r/\sigma'^2-1}\cdot\Phi\left(\frac{-\log d_t - \left(r-\sigma'^2/2\right)\cdot(T-t)}{\sigma'\cdot\sqrt{T-t}}\right)\right)}{\Phi\left(\frac{-\log d_t + \left(r+\sigma'^2/2\right)\cdot(T-t)}{\sigma'\cdot\sqrt{T-t}}\right) + d_t^{2\cdot r/\sigma'^2+1}\cdot\Phi\left(\frac{-\log d_t - \left(r+\sigma'^2/2\right)\cdot(T-t)}{\sigma'\cdot\sqrt{T-t}}\right)} - 1} \tag{6.2}$$

where

$$d_t = \left(\frac{S_t}{L'}\right)^{\gamma} = \left(1 + \frac{\text{div}_t}{\text{int}}\right)^{\gamma} \tag{6.3}$$

can be defined in terms of the dividend and interest payment rates;

$$\sigma' = \sigma \cdot \gamma \tag{6.4}$$

is shorthand for the effective volatility of this quantity; and

$$s_t = \frac{\text{int}}{D_t} - r \tag{6.5}$$

is the perpetual spread introduced in (2.5) expressed here in terms of the interest payment rate and the aggregate debt capitalization.

A consequence of the accessible default is that

$$\lim_{T\to t} s_{\text{swap}}(t,T) = 0$$

That is, the firm should be able to borrow at a rate close to risk-free for very short term loans such as commercial paper.

For very long-term swaps, I can show that

$$\lim_{T\to\infty} s_{\text{swap}}(t,T) = s_t$$

Since the perpetual spread does not depend on L', long-term spreads are evidently a function of ultimate liquidation, not interim financial distress.

[6]This has been corrected from the 2006 version following the comments of Simon Babbs.

Furthermore, I can show that

$$\exists \quad T' > t \quad \ni \quad \frac{\partial}{\partial T} s_{\text{swap}}(t,T) < 0 \quad \forall \quad T > T'$$

Since the spread is positive and continuous in term, it must climb to some peak value and then converge to the asymptote from above; i.e. the spread term structure is generally "humped".

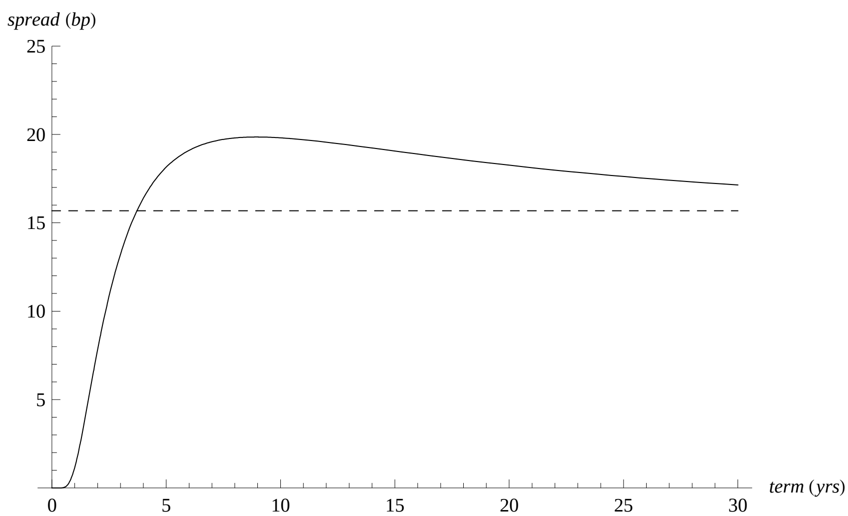

Figure 4: CDS spread and asymptotic level for $S_0 = 100$ with $r = 0.05$, $\delta = 0.03$, $\sigma = 0.2$, and $L = 20$.

6.2 Equity Options

We can use the results from Appendix D.2 to value European-style derivatives on the equity value.

For example, consider a call expiring at T struck at X. The value of this for $t < T$ is

$$e^{-r\cdot(T-t)} \cdot \tilde{\mathsf{E}}_t \left\{ \left(L \cdot f_\gamma \left(e^{\sigma\cdot(\tilde{B}'_{T-t} - M)} \right) - X \right) \cdot H \left(\tilde{B}'_{T-t} - B \right) \right\} \tag{6.6}$$

where

$$B = M + \frac{\log f_\gamma^{-1}(X/L)}{\sigma}$$

$$M = -\frac{\log f_\gamma^{-1}(E_t/L)}{\sigma}$$

and

$$\theta = \frac{\gamma\cdot\sigma}{2} - \frac{r}{\gamma\cdot\sigma}$$

in the risk-neutral version. Since $f_\gamma(\cdot)$ is a linear combination of powers of the argument, we can use (D.16) to write down the following result.

$$\begin{aligned}
&e^{-r\cdot(T-t)} \cdot \tilde{\mathsf{E}}_t \left\{\max\left(E_T - X, 0\right)\right\} = \\
&\quad L \cdot e^{-r\cdot(T-t)} \cdot \Bigg\{ e^{(\theta+\sigma/2)\cdot\sigma\cdot(T-t)-\sigma\cdot M} \cdot \Phi\left(\frac{-B + (\theta+\sigma)\cdot(T-t)}{\sqrt{T-t}}\right) \\
&\qquad - e^{(\theta+\sigma/2)\cdot(\sigma\cdot(T-t)+2\cdot M)} \cdot \Phi\left(\frac{2\cdot M - B + (\theta+\sigma)\cdot(T-t)}{\sqrt{T-t}}\right) \\
&\qquad - \left(e^{\sigma\cdot(B-M)} + \frac{1}{\gamma}\cdot e^{-\gamma\cdot\sigma\cdot(B-M)}\right) \cdot \Phi\left(\frac{-B + \theta\cdot(T-t)}{\sqrt{T-t}}\right) \\
&+ \left(e^{\sigma\cdot(B-M)+2\cdot\theta\cdot M} + \frac{1}{\gamma}\cdot e^{-\gamma\cdot\sigma\cdot(B-M)+2\cdot\theta\cdot M}\right) \cdot \Phi\left(\frac{2\cdot M - B + \theta\cdot(T-t)}{\sqrt{T-t}}\right) \\
&\qquad + \frac{1}{\gamma}\cdot e^{r\cdot(T-t)+\gamma\cdot\sigma\cdot M} \cdot \Phi\left(\frac{-B + (\theta-\gamma\cdot\sigma)\cdot(T-t)}{\sqrt{T-t}}\right) \\
&\quad -\frac{1}{\gamma}\cdot e^{r\cdot(T-t)+(2\cdot\theta-\gamma\cdot\sigma)\cdot M} \cdot \Phi\left(\frac{2\cdot M - B + (\theta-\gamma\cdot\sigma)\cdot(T-t)}{\sqrt{T-t}}\right)\Bigg\}
\end{aligned} \tag{6.7}$$

In comparing this to the Black-Scholes formula, note that the equivalent instantaneous dividend yield is

$$\delta_t' = \frac{\delta \cdot f_\gamma^{-1}\left(E_t/L\right) - r \cdot \left(1 + \frac{1}{\gamma}\right)}{E_t/L} \tag{6.8}$$

The implied volatility here is a function of the level of the equity, the strike price and term of the option, and the parameters of the model. As a baseline, we know that the instantaneous volatility of equity is

$$\sigma_t' = \sigma \cdot \Omega_t \tag{6.9}$$

The implied volatility of near-the-money options is generally close to the instantaneous equity volatility. The downward slope to the curve is a typical pattern observed in practice.

It is also worth noting that this model gives a dependence between implied volatility and the level of the underlying equity value. This pattern has been termed the leverage effect.

From the observed market price of derivatives, the asset volatility σ can be implied.

It is important to contrast the preceding analysis with the compound option approach taken by Geske (1979) where the default option is assumed to have fixed term and European-style exercise. Furthermore, modeling the equity value as an option premium obscures the fundamental role of dividends.

7 Conclusion

The structural model with perpetual debt is surprisingly tractable, yielding consistent exact results for several important classes of securities, include eq-

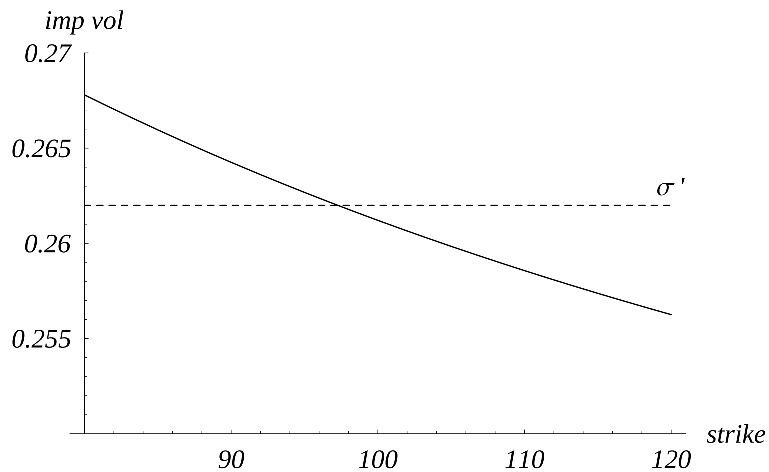

Figure 5: Implied Black-Scholes volatility for a one-year option with $E_0 = 100$, $r = 0.05$, $\delta = 0.03$, $\sigma = 0.2$, and $L = 20$.

uity, corporate debt, equity options, and default swaps.

Three key mathematical observations lead to these results: First, careful consideration of liability cashflows allows one to construct an arbitrage-free model with no explicit time dependence; second, relevant quantities in this setting are simple functions of a stopped drifted brownian motion; and third, evaluating expectations (under the risk-neutral or the real-world measure) of functions of a stopped drifted brownian motion is straight-forward. These observations are familiar to students of the analytical treatment of exotic options. Their combination into a coherent structural model is, I believe, novel and valuable.

I expect this framework to become the basis for a more consistent treatment of equity and debt securities, and ultimately lead to further illumination of the underlying economic determinants of investment risk and value.

A Calibration Example

Calibrating the model entails identifying values for the following parameters and levels

r	risk-free interest rate
σ	asset volatility rate
δ	asset payout rate
μ	asset growth rate
K	asset put strike
S_0	enterprise value

Here I will describe one possible method to achieve this, but others might be more suitable. The objective, of course, is to identify values for the parameters that are as stable as possible, and values for the levels that are as current and accurate as possible.

A.1 Risk-Free Interest

The risk-free rate has several roles in this model, but its most important use is in valuing perpetuities. Therefore, we are interested in a stable estimate for the long-term risk-free rate.

As an alternative to a complete bootstrap, let us focus on the bellwether ten- and thirty-year government obligations and note the following fact about continuous par yields, y_T.

$$\frac{1 - e^{-30 \cdot r_{30}}}{y_{30}} - \frac{1 - e^{-10 \cdot r_{10}}}{y_{10}} = \int_{10}^{30} e^{-t \cdot r_t} \, dt \tag{A.1}$$

Let us assume that r is the constant value of r_t that solves this. This leads to the specification

$$\frac{1}{y_{30}} + \left(\frac{1}{r} - \frac{1}{y_{30}} \right) \cdot e^{-30 \cdot r} = \frac{1}{y_{10}} + \left(\frac{1}{r} - \frac{1}{y_{10}} \right) \cdot e^{-10 \cdot r} \tag{A.2}$$

which can be solved numerically for r.

For example, on May 31, 2006, the conventional yield on the T 5-1/8 5/16 was 5.121%, and the conventional yield on the T 4-1/2 2/36 was 5.229%. Converting from semi-annual to continuous compounding gives $y_{10} = 0.0506$ and $y_{30} = 0.0516$ and solving (A.2) gives $r = 0.0528$ per year.

A.2 Asset Put Strike

Let us use IBM as our example. In 1Q06, IBM reported interest expense of 66 million dollars, which we can annualize to get a projected 264 million dollars for all of 2006. Since the annual interest expense in our model is $r \cdot K$, we can infer that $K = 5,001$ million dollars.

A.3 Market Capitalization

Based on the May 31 closing price of 79.90 per share and the most recent count of 1,550.395 million shares outstanding, we know that the market capitalization of IBM is $E_0 = 123,877$ million dollars.

A.4 Asset Payout Rate

IBM's most recently declared quarterly dividend was 30 cents per share. This means that $\delta \cdot S_0 = 2,124$ million dollars per year including interest payments.

If we can assume that γ is sufficiently large (which we can verify later), then we can use the limit

$$\lim_{\gamma \to \infty} S_0 = E_0 + K \tag{A.3}$$

to determine that $\delta = 0.0165$ per year, approximately.

A.5 Asset Volatility Rate

If we can assume that γ is sufficiently large, then we can use the limit

$$\lim_{\gamma \to \infty} \Omega_0 = 1 + \frac{K}{E_0} \tag{A.4}$$

and the observation that the implied volatility on near-the-money options is close to $\sigma \cdot \Omega_0$ for all terms to expiration.

For example, the Jan '08 100 calls on IBM closed with an implied volatility of about 18.1%. Dividing by the asymptotic equity elasticity gives $\sigma = 0.175$ per year, approximately.

Combining r, δ, and σ, we get that $\gamma = 2.67$, and applying this to the asset put strike, we get that $L = 3,638$ million dollars.

A.6 Enterprise Value

Now that we have γ, we do not need to depend further on approximations. We can use the relationship

$$S_0 = L \cdot f_\gamma^{-1}\left(\frac{E_0}{L}\right) \tag{A.5}$$

to derive the enterprise value. In this case, we get $S_0 = 128,877$ million dollars, which is very close indeed to the approximate value above in (A.3), confirming the assumption.

A.7 Spreads

For May 31, 2006, the five- and ten-year indicative CDS spreads for IBM were 15 and 34 basis points respectively. These values are significantly higher than the values given by (6.2) based on the prior calibration. In fact, these spreads are consistent with an asset volatility level closer to $\sigma = 0.38$.

This inconsistency is common to all structural models. Empirical work on shocks and stochastic volatility such as Zhang, Zhou & Zhu (2005) may offer a way forward towards resolving this.

A.8 Asset Growth Rate

The *cum div* asset growth rate, μ, is not necessary for any valuations, but it is part of the description of horizon values. Various approaches could be taken to estimate this, including application of the capital asset pricing model, analysis of the growth rate of reported accounting earnings, or interpretation of obligor credit ratings. I will not go into this here.

B Black-Scholes

To avoid arbitrage, any asset that produces dividends at a rate δ and has a value V that depends on time t and the values V_i of other assets that produce dividends δ_i must satisfy the following version of the Black-Scholes equation,

$$\frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial V_i} \cdot V_i \cdot (r - \delta_i) + \tfrac{1}{2} \cdot \sum_{i,j} \frac{\partial^2 V}{\partial V_i \, \partial V_j} \cdot V_i \cdot V_j \cdot \sigma_{i,j} = V \cdot (r - \delta) \quad \text{(B.1)}$$

for risk-free interest r and asset return covariance $\sigma_{i,j}$.

For the perpetual American-style put, p, there is no dividend or explicit time dependence, and the dependent security is S which produces a dividend $\delta \cdot S$ and has volatility σ. The Black-Scholes equation is

$$\frac{\partial p}{\partial S} \cdot S \cdot (r - \delta) + \tfrac{1}{2} \cdot \frac{\partial^2 p}{\partial S^2} \cdot S^2 \cdot \sigma^2 = p \cdot r \quad \text{(B.2)}$$

We can verify by substituting in the solution (2.1) that this is satisfied provided

$$-\gamma \cdot (r - \delta) + \tfrac{1}{2} \cdot \gamma \cdot (\gamma + 1) \cdot \sigma^2 = r \quad \text{(B.3)}$$

and that this in turn is satisfied by the definition of γ in (2.2).

The debt, $D = K - p$, produces a dividend $r \cdot K$ and has two underlyings, S and K, which produce dividends $\delta \cdot S$ and $r \cdot K$. The Black-Scholes equation

$$-\frac{\partial p}{\partial S} \cdot S \cdot (r - \delta) - \tfrac{1}{2} \cdot \frac{\partial^2 p}{\partial S^2} \cdot S^2 \cdot \sigma^2 = D \cdot r - K \cdot r \quad \text{(B.4)}$$

is confirmed by noticing that both sides are equal to $-r \cdot p$.

The equity, $E = S - K + p$, has the same two underlings, S and K. We can verify that the Black-Scholes equation is satisfied if the dividend equals $\delta \cdot S - r \cdot K$.

C Moments

We can use the result (D.10) to evaluate moments of horizon values of equity and debt.

Consider the definition of $f_\gamma(\cdot)$ in (4.2) upon which the definitions of the equity and debt values are based. Natural powers of this are linear combinations of terms involving powers of the argument.

$$f_\gamma(x)^N = \sum_{i=0}^{N}\sum_{j=0}^{N-i}(-1)^{N-i-j}\cdot\binom{N}{i}\cdot\binom{N-i}{j}\cdot\frac{(1+\gamma)^{N-i-j}}{\gamma^{N-i}}\cdot x^{i-\gamma\cdot j} \quad \text{(C.1)}$$

The horizon value of the shareholders' equity is defined in terms of $f_\gamma(\cdot)$ in (4.1). Using (D.10), we can write down an expression for the N-th moment of E_T.

$$\begin{aligned}\mathsf{E}_0\left\{E_T^N\right\} = L^N\cdot\sum_{i=0}^{N}\sum_{j=0}^{N-i}(-1)^{N-i-j}\cdot\binom{N}{i}\cdot\binom{N-i}{j}\\ \cdot\frac{(1+\gamma)^{N-i-j}}{\gamma^{N-i}}\cdot e^{(i-\gamma\cdot j)\cdot\left(\mu-\delta+(i-\gamma\cdot j-1)\cdot\sigma^2/2\right)\cdot T}\\ \cdot\left(\left(\frac{S_0}{L}\right)^{i-\gamma\cdot j}\cdot\Phi\left(\frac{\left(\mu-\delta+\left(i-\gamma\cdot j-\frac{1}{2}\right)\cdot\sigma^2\right)\cdot T+\log S_0/L}{\sigma\cdot\sqrt{T}}\right)\right.\\ \left.-\left(\frac{S_0}{L}\right)^{1-2\cdot\frac{\mu-\delta}{\sigma^2}}\cdot\Phi\left(\frac{\left(\mu-\delta+\left(i-\gamma\cdot j-\frac{1}{2}\right)\cdot\sigma^2\right)\cdot T-\log S_0/L}{\sigma\cdot\sqrt{T}}\right)\right)\end{aligned} \quad \text{(C.2)}$$

Similarly, using $x - f_\gamma(x)$ in place of $f_\gamma(x)$, we see that the N-th moment of the horizon value of the debt is

$$\begin{aligned}\mathsf{E}_0\left\{D_T^N\right\} = L^N\cdot\sum_{i=0}^{N}(-1)^{i}\cdot\binom{N}{i}\\ \cdot\frac{(1+\gamma)^{N-i}}{\gamma^{N}}\cdot e^{-\gamma\cdot i\cdot\left(\mu-\delta-(\gamma\cdot i+1)\cdot\sigma^2/2\right)\cdot T}\\ \cdot\left(\left(\frac{S_0}{L}\right)^{-\gamma\cdot i}\cdot\Phi\left(\frac{\left(\mu-\delta-\left(\gamma\cdot i+\frac{1}{2}\right)\cdot\sigma^2\right)\cdot T+\log S_0/L}{\sigma\cdot\sqrt{T}}\right)\right.\\ \left.-\left(\frac{S_0}{L}\right)^{1-2\cdot\frac{\mu-\delta}{\sigma^2}}\cdot\Phi\left(\frac{\left(\mu-\delta-\left(\gamma\cdot i+\frac{1}{2}\right)\cdot\sigma^2\right)\cdot T-\log S_0/L}{\sigma\cdot\sqrt{T}}\right)\right)\end{aligned} \quad \text{(C.3)}$$

For short horizons, we can use the approximation in (D.11). Some central moments to lowest order in T are

$$\begin{aligned}\mathsf{E}_0\left\{E_T\right\}\approx\\ E_0+L\cdot\left(\frac{S_0}{L}\right)^{-\gamma}\cdot\left((\mu-\delta)\cdot\left(\left(\frac{S_0}{L}\right)^{1+\gamma}-1\right)+\frac{1}{2}\cdot\sigma^2\cdot(1+\gamma)\right)\cdot T\end{aligned} \quad \text{(C.4a)}$$

$$\mathsf{var}_0\left\{E_T\right\} \approx L^2 \cdot \left(\frac{S_0}{L}\right)^{-2\cdot\gamma} \cdot \left(\left(\frac{S_0}{L}\right)^{1+\gamma} - 1\right)^2 \cdot \sigma^2 \cdot T \tag{C.4b}$$

$$\mathsf{skew}_0\left\{E_T\right\} \approx 3 \cdot \left(1 + \frac{1+\gamma}{\left(\frac{S_0}{L}\right)^{1+\gamma} - 1}\right) \cdot \sigma \cdot \sqrt{T} \tag{C.4c}$$

$$\mathsf{kurt}_0\left\{E_T\right\} \approx 3 + 16 \cdot \left(1 + (1+\gamma) \cdot \frac{\frac{7-\gamma}{4} \cdot \left(\frac{S_0}{L}\right)^{1+\gamma} + \gamma - 1}{\left(\left(\frac{S_0}{L}\right)^{1+\gamma} - 1\right)^2}\right) \cdot \sigma^2 \cdot T \tag{C.4d}$$

$$\mathsf{E}_0\left\{D_T\right\} \approx D_0 + L \cdot \left(\frac{S_0}{L}\right)^{-\gamma} \cdot \big(\mu - \delta - \tfrac{1}{2} \cdot \sigma^2 \cdot (1+\gamma)\big) \cdot T \tag{C.5a}$$

$$\mathsf{var}_0\left\{D_T\right\} \approx L^2 \cdot \left(\frac{S_0}{L}\right)^{-2\cdot\gamma} \cdot \sigma^2 \cdot T \tag{C.5b}$$

$$\mathsf{skew}_0\left\{D_T\right\} \approx -3 \cdot \gamma \cdot \sigma \cdot \sqrt{T} \tag{C.5c}$$

$$\mathsf{kurt}_0\left\{D_T\right\} \approx 3 + 16 \cdot \gamma^2 \cdot \sigma^2 \cdot T \tag{C.5d}$$

For comparison, these are the low-order central moments of the log-normal assets for small T.

$$\mathsf{E}_0\left\{S_T\right\} \approx S_0 + S_0 \cdot (\mu - \delta) \cdot T \tag{C.6a}$$

$$\mathsf{var}_0\left\{S_T\right\} \approx S_0^2 \cdot \sigma^2 \cdot T \tag{C.6b}$$

$$\mathsf{skew}_0\left\{S_T\right\} \approx 3 \cdot \sigma \cdot \sqrt{T} \tag{C.6c}$$

$$\mathsf{kurt}_0\left\{S_T\right\} \approx 3 + 16 \cdot \sigma^2 \cdot T \tag{C.6d}$$

D Excursion of a Brownian Motion

For background, see Karatzas & Shreve (1998).

D.1 Joint Minimum and Terminal Values

Consider a brownian motion B_t. Define

$$M_T = \min_{0<t<T} B_t \tag{D.1}$$

From the reflection principle, we know

$$\mathsf{Pr}_0\left\{M_T < m\ ,\ B_T > b\right\} = \mathsf{Pr}_0\left\{B_T < 2\cdot m - b\right\} \tag{D.2}$$

Since $B_T \sim \mathrm{N}(0, T)$, we know

$$\begin{aligned}
&\mathsf{Pr}_0\left\{M_T \in dm\ ,\ B_T \in db\right\} \\
&\qquad = -\frac{\partial^2}{\partial b\,\partial m}\left(\frac{1}{\sqrt{2\pi\cdot T}}\int_{-\infty}^{2\cdot m-b} e^{-z^2/(2\cdot T)}\,dz\right) db\,dm \\
&\qquad = 2\cdot\frac{b-2\cdot m}{T}\cdot\frac{e^{-(b-2\cdot m)^2/(2\cdot T)}}{\sqrt{2\pi\cdot T}}\,db\,dm
\end{aligned} \tag{D.3}$$

for $m < 0$ and $b > m$.

Define a new stochastic process $\tilde{B}_t$ by adding a drift term.

$$\tilde{B}_t = B_t + \theta\cdot t \tag{D.4}$$

Any probabilities associated with $\tilde{B}$ can be expressed in terms of probabilities associated with B and the Radon-Nikodym change of measure.

$$Z\left(T, b\right) = \frac{\mathsf{Pr}_0\left\{B_T \in db\right\}}{\mathsf{Pr}_0\left\{\tilde{B}_T \in db\right\}} = \frac{e^{-b^2/(2\cdot T)}}{\sqrt{2\pi\cdot T}}\Big/\frac{e^{-(b-\theta\cdot T)^2/(2\cdot T)}}{\sqrt{2\pi\cdot T}} = e^{-\theta\cdot b+\frac{\theta^2\cdot T}{2}} \tag{D.5}$$

Using this, we know

$$\begin{aligned}
&\mathsf{Pr}_0\left\{\tilde{M}_T \in dm\ ,\ \tilde{B}_T \in db\right\} \\
&\qquad = \frac{\mathsf{Pr}_0\left\{M_T \in dm\ ,\ B_T \in db\right\}}{Z(T, b)}\,db\,dm \\
&\qquad = 2\cdot e^{2\cdot\theta\cdot m}\cdot\frac{b-2\cdot m}{T}\cdot\frac{e^{-(b-2\cdot m-\theta\cdot T)^2/(2\cdot T)}}{\sqrt{2\pi\cdot T}}\,db\,dm
\end{aligned} \tag{D.6}$$

in the original measure.

So the expectation of any function that depends on the minimum and final value of a drifted brownian motion is

$$\mathsf{E}_0\left\{f\left(\min_{0<t<T}\left[B_t+\theta\cdot t\right],B_T+\theta\cdot T\right)\right\}$$
$$=\int_{-\infty}^{0}\int_{m}^{\infty} f(m,b)\cdot 2\cdot e^{2\cdot\theta\cdot m}\cdot\frac{b-2\cdot m}{T}\cdot\frac{e^{-(b-2\cdot m-\theta\cdot T)^2/(2\cdot T)}}{\sqrt{2\pi\cdot T}}\,db\,dm \quad \text{(D.7)}$$

For example, to calculate the marginal probability that the minimum value attained between time zero and T falls below some threshold $M<0$, we can evaluate the integral above with

$$f(m,b)=H(M-m)$$

where $H(\cdot)$ is the step function.

This yields

$$\mathsf{Pr}_0\left\{\min_{0<t<T}\left[B_t+\theta\cdot t\right]<M\right\}=e^{2\cdot\theta\cdot M}\cdot\Phi\left(\frac{M+\theta\cdot T}{\sqrt{T}}\right)+\Phi\left(\frac{M-\theta\cdot T}{\sqrt{T}}\right) \quad \text{(D.8)}$$

where $\Phi(\cdot)$ is the standard normal CDF.

Note that this probability goes to zero as $T\to 0$. In fact, application of L'Hôpital's Rule reveals that $q_T(S_0)$ grows slower than any power of T.

Also, for $T\to\infty$, the limit of the probability is

$$\mathsf{Pr}_0\left\{\min_{t>0}\left[B_t+\theta\cdot t\right]<M\right\}=\begin{cases}e^{2\cdot\theta\cdot M} & \theta>0\\ 1 & \text{otherwise}\end{cases} \quad \text{(D.9)}$$

Another case is

$$f(m,b)=H(m-M)\cdot e^{b\cdot N}$$

which evaluates to

$$\mathsf{E}_0\left\{H\left(\min_{0<t<T}\left[B_t+\theta\cdot t\right]-M\right)\cdot e^{(B_T+\theta\cdot T)\cdot N}\right\}=e^{N\cdot(\theta+N/2)\cdot T}$$
$$\cdot\left(\Phi\left(\frac{(\theta+N)\cdot T-M}{\sqrt{T}}\right)-e^{2\cdot(\theta+N)\cdot M}\cdot\Phi\left(\frac{(\theta+N)\cdot T+M}{\sqrt{T}}\right)\right) \quad \text{(D.10)}$$

For small T, this is approximately

$$\mathsf{E}_0\left\{H\left(\min_{0<t<T}\left[B_t+\theta\cdot t\right]-M\right)\cdot e^{(B_T+\theta\cdot T)\cdot N}\right\}\approx e^{N\cdot(\theta+N/2)\cdot T} \quad \text{(D.11)}$$

consistent with a log-normal random variable.

D.2 Terminal Value

We are interested in the distribution of the terminal value of a drifted brownian motion subject to an absorbing lower boundary. To calculate the probability that the terminal value does not exceed a given level, we can evaluate the expectation in (D.7) with

$$f(m,b) = H(M-m) + H(m-M) \cdot H(B-b) \tag{D.12}$$

which evaluates to

$$\begin{aligned}\mathsf{Pr}_0 \left\{ \min_{0<t<T} [B_t + \theta \cdot t] < M \vee \left(\min_{0<t<T} [B_t + \theta \cdot t] > M \wedge B_T + \theta \cdot T < B \right) \right\} \\ = \Phi\left(\frac{B - \theta \cdot T}{\sqrt{T}}\right) + e^{2\cdot\theta\cdot M} \cdot \Phi\left(\frac{2 \cdot M - B + \theta \cdot T}{\sqrt{T}}\right)\end{aligned} \tag{D.13}$$

for $B \geq M$.

Let us refer to the drifted and stopped brownian motion by the symbol $\tilde{B}'_t$.

$$\tilde{B}'_T = \begin{cases} B_T + \theta \cdot T & \min_{0<t<T} [B_t + \theta \cdot t] > M \\ M & \text{otherwise} \end{cases} \tag{D.14}$$

Differentiating, we see that the density of this final value is

$$\mathsf{Pr}_0 \left\{ \tilde{B}'_T \in db \right\} = \begin{cases} \left(1 - e^{2\cdot M\cdot(b-M)/T}\right) \cdot e^{-(b-\theta\cdot T)^2/(2\cdot T)} / \sqrt{2\pi \cdot T}\, db & b > M \\ e^{2\cdot\theta\cdot M} \cdot \Phi\left(\frac{M+\theta\cdot T}{\sqrt{T}}\right) + \Phi\left(\frac{M-\theta\cdot T}{\sqrt{T}}\right) & b = M \\ 0 & b < M \end{cases} \tag{D.15}$$

Note that for values much greater than M, the density of $\tilde{B}'_T$ is approximately normal.

The pole at M is a non-essential singularity.

In order to value European-style contingent claims on equity and debt, it will be useful to use the following result.

$$\begin{aligned}\mathsf{E}_0 \left\{ e^{\tilde{B}'_T \cdot N} \cdot H\left(\tilde{B}'_T - B\right) \right\} = e^{N\cdot(\theta+N/2)\cdot T} \\ \cdot \left(\Phi\left(\frac{-B + (\theta+N) \cdot T}{\sqrt{T}}\right) - e^{2\cdot(\theta+N)\cdot M} \cdot \Phi\left(\frac{2 \cdot M - B + (\theta+N) \cdot T}{\sqrt{T}}\right)\right)\end{aligned} \tag{D.16}$$

for $B \geq M$ and any N.

D.3 Passage Time

To find the distribution of the passage time τ_M corresponding to some lower level M, we can again use the reflection principal for an un-drifted brownian motion.

$$\mathsf{Pr}_0 \{\tau_M < t\} = 2 \cdot \mathsf{Pr}_0 \{B_t < M\} = \frac{2}{\sqrt{2\pi \cdot t}} \int_{-\infty}^{M} e^{-b^2/(2\cdot t)}\, db \tag{D.17}$$

or

$$\mathsf{Pr}_0 \left\{ \tau_M \in dt \right\} = \frac{-M}{\sqrt{2\pi \cdot t^3}} \cdot e^{-M^2/(2\cdot t)} \, dt \tag{D.18}$$

for $T > 0$ and $M < 0$.

To incorporate a drift θ, we can apply the same change of measure as before.

$$\begin{aligned} \mathsf{Pr}_0 \left\{ \tau_M^\theta \in dt \right\} &= \frac{-M}{\sqrt{2\pi \cdot t^3}} \cdot e^{-M^2/(2\cdot t)} \cdot e^{\theta \cdot M - \theta^2 \cdot t/2} \, dt \\ &= \frac{-M}{\sqrt{2\pi \cdot t^3}} \cdot e^{-(\theta\cdot t - M)^2/(2\cdot t)} \, dt \end{aligned} \tag{D.19}$$

We will be interested in evaluating expectations of the following form.

$$\begin{aligned} &\mathsf{E}_0 \left\{ e^{-\alpha \cdot \tau_M^\theta} \cdot H(T - \tau_M^\theta) \right\} = \\ &\qquad e^{M \cdot \left(\theta + \sqrt{\theta^2 + 2\cdot\alpha}\right)} \cdot \Phi\left(\frac{M + T \cdot \sqrt{\theta^2 + 2 \cdot \alpha}}{\sqrt{T}} \right) \\ &\qquad\quad + e^{M \cdot \left(\theta - \sqrt{\theta^2 + 2\cdot\alpha}\right)} \cdot \Phi\left(\frac{M - T \cdot \sqrt{\theta^2 + 2 \cdot \alpha}}{\sqrt{T}} \right) \end{aligned} \tag{D.20}$$

Letting $\alpha = 0$ and $T \to \infty$, we see that $\mathsf{Pr}_0 \left\{ \tau_M^\theta < \infty \right\}$ is consistent with (D.9). In particular, if $\theta > 0$, then this probability is less than unity, indicating that there is a chance that the threshold is never breached.

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