Attilio Meucci

Lehman Brothers, Inc., New York

personal website: **symmys.com**

Issues in Quantitative Portfolio Management: Handling Estimation Risk

AGENDA

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Estimation vs. Modeling

Classical Optimization and Estimation Risk

Black-Litterman Optimization

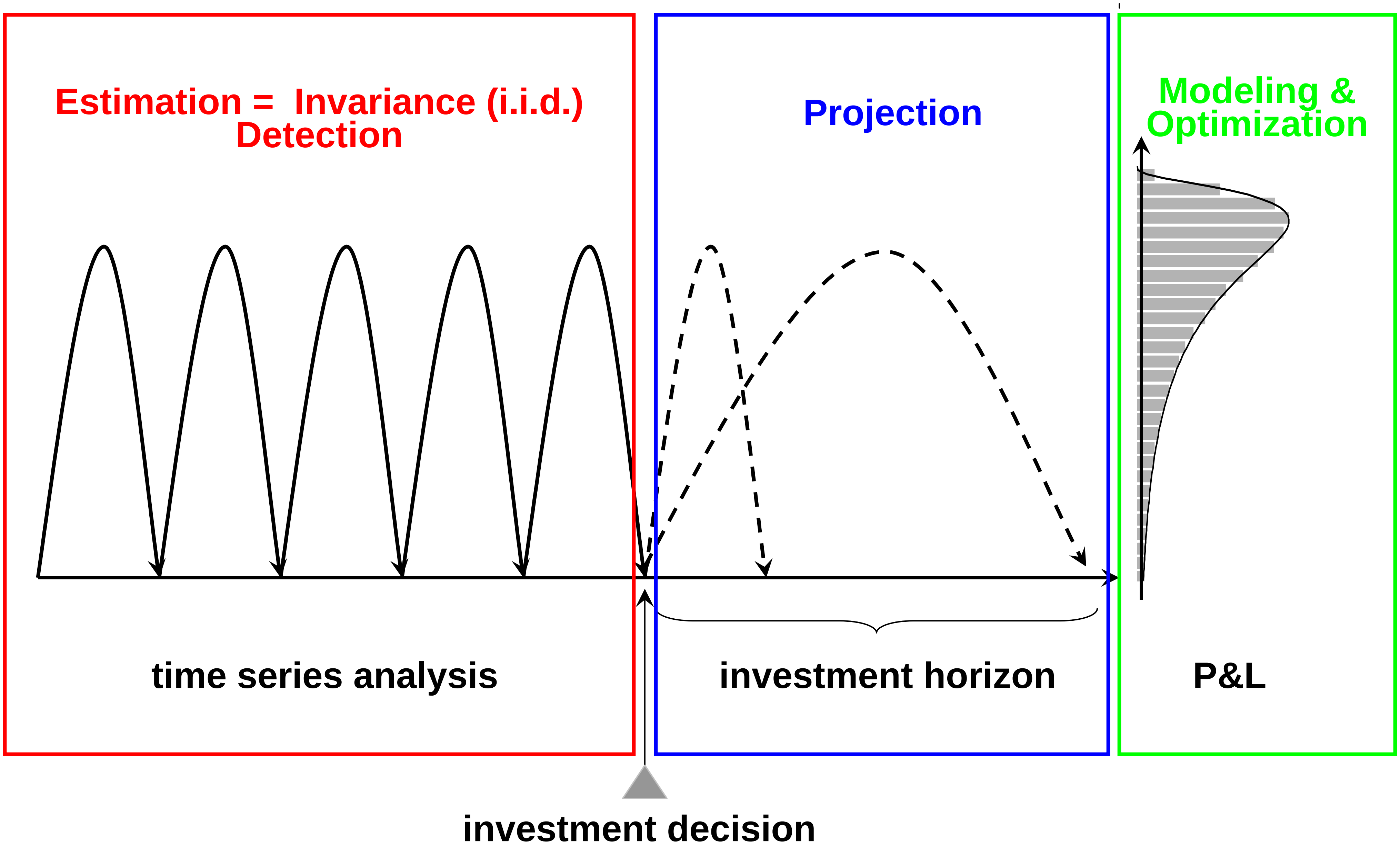
Robust Optimization

Bayesian Optimization

Robust Bayesian Optimization

References

ESTIMATION vs. MODELING – general conceptual framework



ESTIMATION vs. MODELING – fixed-income PCA trading recipe

1. consider N series of T observations of homogeneous forward rates

   $\boldsymbol{X}$ (TxN panel)

2. define $\boldsymbol{S} \equiv \boldsymbol{X}'\boldsymbol{X}$ (NxN positive definite matrix)

3. run PCA $\boldsymbol{S} \equiv \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}'$ (eigenvectors-eigenvalues-eigenvectors)

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4. analyze the series $y \equiv \boldsymbol{X}\boldsymbol{e}^{(N)}$ of the last factor

   - z-score: structural bands
   - "juice": b.p. from mean
   - roll-down/slide-adjusted prospective Sharpe ratio
   - reversion timeframe

   } "small picture"

   - market events (e.g. Fed, Thursday "numbers",…)
   - relation with other series (e.g. oil prices)

   } "big picture"

5. convert basis points to PnL/risk exposure by dv01

variations: transform series, include mean, support series (PCA-regression),…

ESTIMATION vs. MODELING – fixed-income PCA trading recipe

estimation

projection

modeling

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5. convert basis points to PnL/risk exposure by dv01

- estimation (backward-looking) and projection/modeling (forward-looking) overlap
- non-linearities not accounted for

ESTIMATION vs. MODELING – fund of funds <u>flawed</u> management recipe

1. consider N series of T observations of fund prices $\boldsymbol{P}$ (TxN panel)
2. consider the compounded returns $C_{t,n} \equiv \ln\left(P_{t,n}\right) - \ln\left(P_{t-1,n}\right)$
3. estimate covariance (e.g. the sample non-central) $\hat{\Sigma} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{C}_t \mathbf{C}_t{}'$
4. define the expected values (e.g. risk-premium) $\widehat{\mu} \equiv \gamma \operatorname{diag}\left(\hat{\Sigma}\right)$

ESTIMATION vs. MODELING – fund of funds flawed management recipe

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5. solve mean-variance: $\boldsymbol{w}^{(i)} \equiv \underset{\substack{\boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}'\widehat{\Sigma}\boldsymbol{w} \leq v^{(i)}}}{\operatorname{argmax}} \left\{\boldsymbol{w}'\widehat{\mu}\right\}$
   - $\boldsymbol{w} \in \mathcal{C}$ ← investment constraints
   - $\boldsymbol{w}'\widehat{\Sigma}\boldsymbol{w} \leq v^{(i)}$ ← grid of significant variances
6. choose the most suitable combination among $\boldsymbol{w}^{(i)}$ according to preferences

ESTIMATION vs. MODELING – fund of funds flawed management recipe

estimation

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modeling & optimization

5. solve mean-variance: $\boldsymbol{w}^{(i)} \equiv \underset{\substack{\boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}'\widehat{\Sigma}\boldsymbol{w} \leq v^{(i)}}}{\operatorname{argmax}} \left\{\boldsymbol{w}'\widehat{\boldsymbol{\mu}}\right\}$

   investment constraints ($\boldsymbol{w} \in \mathcal{C}$)

   grid of significant variances ($\boldsymbol{w}'\widehat{\Sigma}\boldsymbol{w} \leq v^{(i)}$)

6. choose the most suitable combination among $\boldsymbol{w}^{(i)}$ according to preferences

- estimation (backward-looking) and modeling (forward-looking) overlap
- projection (investment horizon) not accounted for
- non-linearities of compounded returns not accounted for

ESTIMATION vs. MODELING – fund of funds consistent management recipe

- Estimation: compounded returns $C_t^{\tilde{\tau}} \equiv \ln(P_t) - \ln(P_{t-\tilde{\tau}})$ estimation interval

compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion

ESTIMATION vs. MODELING – fund of funds consistent management recipe

- Estimation: compounded returns $C_t^{\tilde{\tau}} \equiv \ln\left(P_t\right) - \ln\left(P_{t-\tilde{\tau}}\right)$

compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion

- Projection to investment horizon $C_t^{\tau} = C_{t-J\tilde{\tau}}^{\tilde{\tau}} + C_{t-(J-1)\tilde{\tau}}^{\tilde{\tau}} + \cdots + C_t^{\tilde{\tau}}$ (investment horizon: τ)

compounded returns can be easily projected to the investment horizon because they are additive (“accordion” expansion)

ESTIMATION vs. MODELING – fund of funds consistent management recipe

- Estimation: compounded returns $C_t^{\tilde{\tau}} \equiv \ln(P_t) - \ln(P_{t-\tilde{\tau}})$

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compounded returns can be easily projected to the investment horizon because they are additive ("accordion" expansion)

- Modeling: linear returns $L_t^{\tau} \equiv P_t / P_{t-\tau} - 1$

linear returns are related to portfolio quantities (P&L): $L_{\Pi} = \boldsymbol{w}'\boldsymbol{L}$

portfolio return (L_{Π}); securities' relative weights ($\boldsymbol{w}$); securities' returns ($\boldsymbol{L}$)

compounded returns are NOT related to portfolio quantities (P&L): $C_{\Pi} \neq \boldsymbol{w}'\boldsymbol{C}$

ESTIMATION vs. MODELING – fund of funds consistent management recipe

- Estimation: compounded returns $C_t^{\tilde{\tau}} \equiv \ln(P_t) - \ln(P_{t-\tilde{\tau}})$

  sample/risk-premium: $\widehat{\Sigma}^{\tilde{\tau}} \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{C}_t^{\tilde{\tau}} \mathbf{C}_t^{\tilde{\tau}}{}' \qquad \widehat{\mu}^{\tilde{\tau}} \equiv \gamma \,\text{diag}\left(\widehat{\Sigma}^{\tilde{\tau}}\right)$

- Projection to investment horizon $C_t^{\tau} \equiv C_{t-J\tilde{\tau}}^{\tilde{\tau}} + C_{t-(J-1)\tilde{\tau}}^{\tilde{\tau}} + \cdots + C_t^{\tilde{\tau}}$

  "square root rule": $\widehat{\Sigma}^{\tau} \equiv \frac{\tau}{\tilde{\tau}} \widehat{\Sigma}^{\tilde{\tau}} \qquad \widehat{\mu}^{\tau} \equiv \frac{\tau}{\tilde{\tau}} \widehat{\mu}^{\tilde{\tau}}$

- Modeling: linear returns $L_t^{\tau} \equiv P_t / P_{t-\tau} - 1$

  Black-Scholes assumption: (log-normal)

$$m_n \equiv E\left\{L_{t,n}^{\tau}\right\} = e^{\frac{\tau}{\tilde{\tau}}\left(\mu_n^{\tau} + \frac{1}{2}\Sigma_{nn}^{\tau}\right)}$$

$$S_{nm} \equiv Cov\left\{L_{t,n}^{\tau}, L_{t,m}^{\tau}\right\} = e^{\frac{\tau}{\tilde{\tau}}\left(\mu_n^{\tau} + \frac{1}{2}\Sigma_{nn}^{\tau} + \mu_m^{\tau} + \frac{1}{2}\Sigma_{mm}^{\tau}\right)}\left(e^{\frac{\tau}{\tilde{\tau}}\Sigma_{nm}^{\tau}} - 1\right)$$

the mean - variance optimization can be fed with the appropriate inputs

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CLASSICAL OPTIMIZATION – mean-variance in theory …

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{m} \right\}$$

$N \times 1$ vector

$$\text{subject to } \begin{cases} \boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \leq v^{(i)} \end{cases}$$

$N \times N$ matrix

$\boldsymbol{w}$: relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $\boldsymbol{w}'\boldsymbol{1} = 1,\ \boldsymbol{w} \geq \boldsymbol{0}$

$v^{(i)}$: significant grid of target variances

$$\boldsymbol{m} \equiv \mathrm{E}\left\{ \boldsymbol{L}_{t+\tau}^{\tau} \right\}$$

$$\boldsymbol{S} \equiv \mathrm{Cov}\left\{ \boldsymbol{L}_{t+\tau}^{\tau} \right\}$$

CLASSICAL OPTIMIZATION – … mean-variance in practice

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{m} \right\}$$

subject to $\begin{cases} \boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \leq v^{(i)} \end{cases}$

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\widehat{\boldsymbol{m}} \right\}$$

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$$\boldsymbol{m} \equiv \mathrm{E}\left\{ \boldsymbol{L}_{t+\tau}^{\tau} \right\}$$

$$\boldsymbol{S} \equiv \mathrm{Cov}\left\{ \boldsymbol{L}_{t+\tau}^{\tau} \right\}$$

$\widehat{\boldsymbol{m}}$: estimate of $\boldsymbol{m}$

e.g. $\widehat{\boldsymbol{m}} \equiv \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{l}_t^{\tau}$

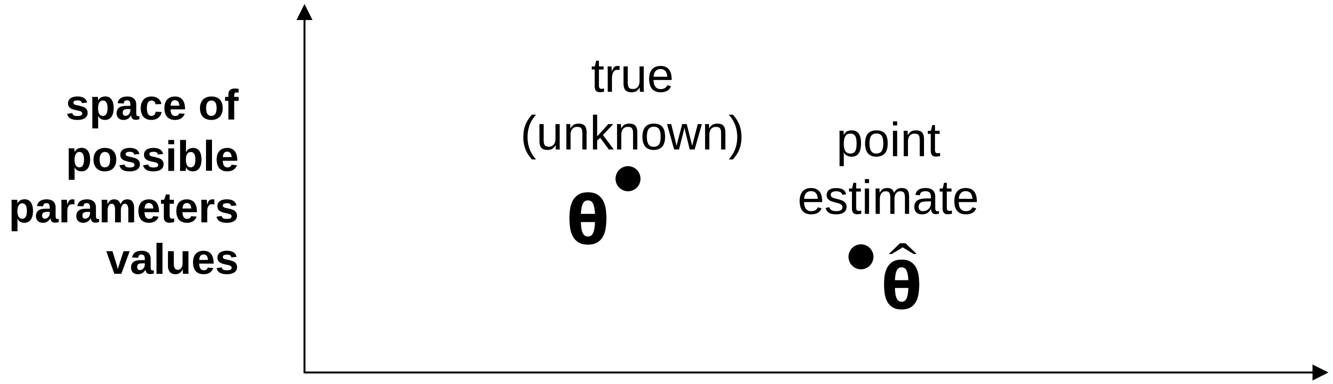
$\widehat{\boldsymbol{S}}$: estimate of $\boldsymbol{S}$

e.g. $\widehat{\boldsymbol{S}} \equiv \frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{l}_t^{\tau} - \widehat{\boldsymbol{m}} \right) \left(\boldsymbol{l}_t^{\tau} - \widehat{\boldsymbol{m}} \right)'$

CLASSICAL OPTIMIZATION – estimation risk

The true optimal allocation is determined by a set of parameters that are estimated with some error:

$$\hat{\boldsymbol{\theta}} \equiv \left(\widehat{m}, \widehat{S} \right) \quad \neq \quad \boldsymbol{\theta} \equiv \left(m, S \right)$$



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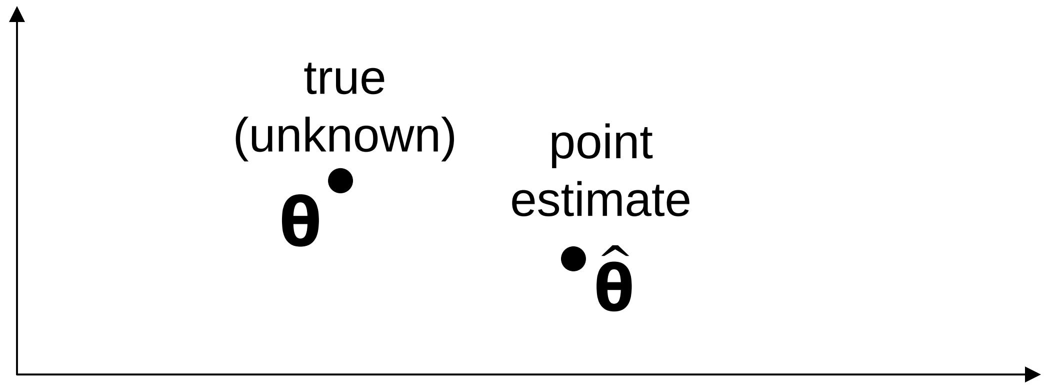
space of possible parameters values



• The classical "optimal" allocation based on point estimates $\hat{\boldsymbol{\theta}} \equiv \left(\widehat{\boldsymbol{m}}, \widehat{\boldsymbol{S}}\right)$ is sub-optimal

• More importantly, the sub-optimality due to estimation error is large (Jobson & Korkie (1980); Best & Grauer (1991); Chopra & Ziemba (1993))

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BLACK-LITTERMAN APPROACH – inputs: prior

linear returns $\boldsymbol{L} \sim \mathrm{N}(\mu, \Sigma)$

unconstrained Markowitz mean-variance optimization

$$\boldsymbol{w}_{\varsigma} \equiv \operatorname{argmax}\left\{\boldsymbol{w}'\mu - \frac{1}{2\varsigma}\boldsymbol{w}'\Sigma\boldsymbol{w}\right\}$$

$$\boldsymbol{w}_{\varsigma} \equiv \varsigma\,\Sigma^{-1}\,\mu \quad \Longleftrightarrow \quad \mu \equiv \frac{1}{\varsigma}\,\Sigma\,\boldsymbol{w}_{\varsigma}$$

well-diversified portfolio (equilibrium - benchmark) → $\boldsymbol{w}_{\varsigma}$

exponentially smoothed estimate of covariance → Σ

average risk propensity ~ 0.4 → ς

market-implied equilibrium prior expected returns → μ

BLACK-LITTERMAN APPROACH – outputs: posterior

“official” prior on linear returns $\boldsymbol{L} \sim \mathrm{N}(\mu, \Sigma)$ ← equilibrium-based estimation

subjective views $\begin{cases} V_1 \equiv \boldsymbol{p}_1 \mu \sim \mathrm{N}(q_1, \omega_1^2) \\ \vdots \\ V_K \equiv \boldsymbol{p}_K \mu \sim \mathrm{N}(q_K, \omega_K^2) \end{cases}$

Bayesian posterior: $\mu_{BL} \equiv \mu + \Sigma \boldsymbol{P}' \left(\boldsymbol{P} \Sigma \boldsymbol{P}' + \Omega \right)^{-1} \left(\boldsymbol{q} - \boldsymbol{P} \mu \right)$

$$\boldsymbol{P} \equiv \begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_K \end{bmatrix}$$

$N \times K$
matrix of stacked
“pick” row-vectors

BLACK-LITTERMAN APPROACH – outputs: portfolios

"official" prior on linear returns $\boldsymbol{L} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ ←----- equilibrium-based estimation

subjective views $\begin{cases} V_1 \equiv \boldsymbol{p}_1 \boldsymbol{\mu} \sim \mathrm{N}\left(q_1, \omega_1^2\right) \\ \vdots \\ V_K \equiv \boldsymbol{p}_K \boldsymbol{\mu} \sim \mathrm{N}\left(q_K, \omega_K^2\right) \end{cases}$

Bayesian posterior: $\boldsymbol{\mu}_{BL} \equiv \boldsymbol{\mu}$ + views

Markowitz mean-variance optimization: $\boldsymbol{w}_{\varsigma} \equiv \underset{\substack{\boldsymbol{w}'\boldsymbol{1} \equiv 1 \\ \boldsymbol{w} \geq \boldsymbol{0}}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{\mu}_{BL} - \frac{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}{2\varsigma} \right\}$

shrinkage to equilibrium

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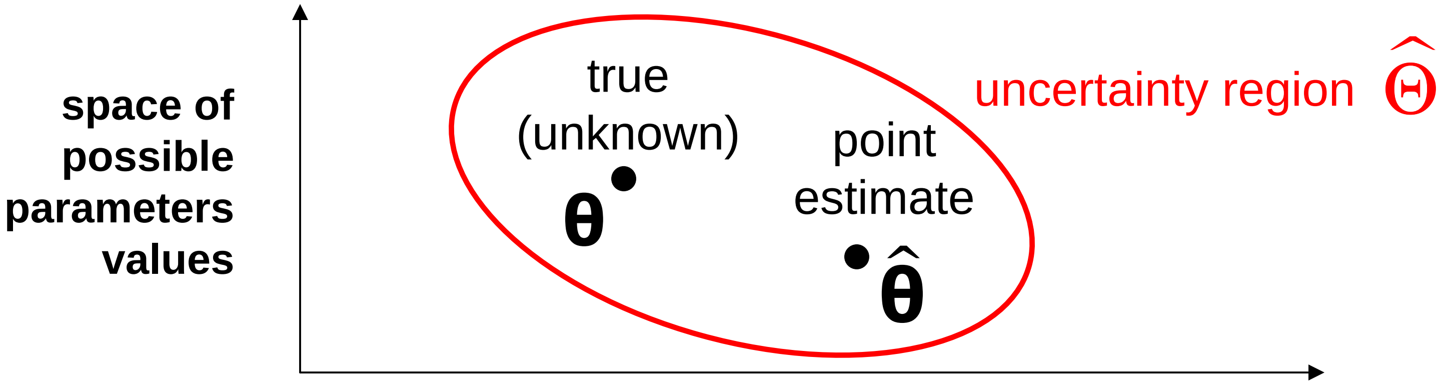
ROBUST OPTIMIZATION – the general framework

• The point estimate for the parameters must be replaced by an uncertainty region that includes the true, unknown parameters:

$$\hat{\boldsymbol{\theta}} \equiv \left(\widehat{\boldsymbol{m}}, \widehat{\boldsymbol{S}}\right) \mapsto \widehat{\Theta}$$



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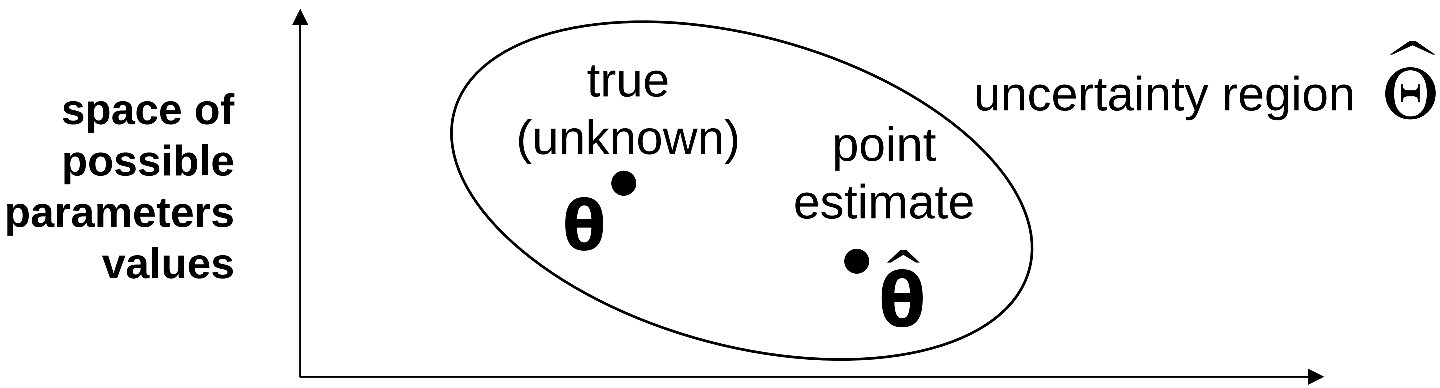


• The allocation optimization must be performed over all the parameters in the uncertainty region:

$$\boldsymbol{w}^{(i)} \equiv \operatorname*{argmax}_{\substack{\boldsymbol{w}\in\mathcal{C} \\ (\boldsymbol{m},\boldsymbol{S})\equiv\left(\widehat{\boldsymbol{m}},\widehat{\boldsymbol{S}}\right)}} \{\ldots\} \quad \mapsto \quad \boldsymbol{w}^{(i)} \equiv \operatorname*{argmax}_{\substack{\boldsymbol{w}\in\mathcal{C} \\ (\boldsymbol{m},\boldsymbol{S})\in\widehat{\Theta}}} \{\ldots\}$$

ROBUST OPTIMIZATION – from the standard mean-variance …

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}' \widehat{\boldsymbol{m}} \right\}$$

$$\text{subject to} \begin{cases} \boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}' \widehat{\boldsymbol{S}} \boldsymbol{w} \leq v^{(i)} \end{cases}$$

$\boldsymbol{w}$: relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $\boldsymbol{w}'\boldsymbol{1} = 1,\ \boldsymbol{w} \geq \boldsymbol{0}$

$v^{(i)}$: significant grid of target variances

$\widehat{\boldsymbol{m}}$: (point) estimate of $\boldsymbol{m}$

$\widehat{\boldsymbol{S}}$: (point) estimate of $\boldsymbol{S}$

ROBUST OPTIMIZATION – … to a conservative mean-variance approach

$$w^{(i)} \equiv \underset{w}{\operatorname{argmax}} \left\{ w' \widehat{m} \right\}$$

$$\text{subject to} \begin{cases} w \in \mathcal{C} \\ w' \widehat{S} w \leq v^{(i)} \end{cases}$$

$$w^{(i)} \equiv \underset{w}{\operatorname{argmax}} \left\{ \min_{m \in \widehat{\Theta}_m} w' m \right\}$$

$$\text{subject to} \begin{cases} w \in \mathcal{C} \\ \max_{S \in \widehat{\Theta}_S} \left\{ w' S w \right\} \leq v^{(i)} \end{cases}$$

w : relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $w'1 = 1, \; w \geq 0$

$v^{(i)}$: significant grid of target variances

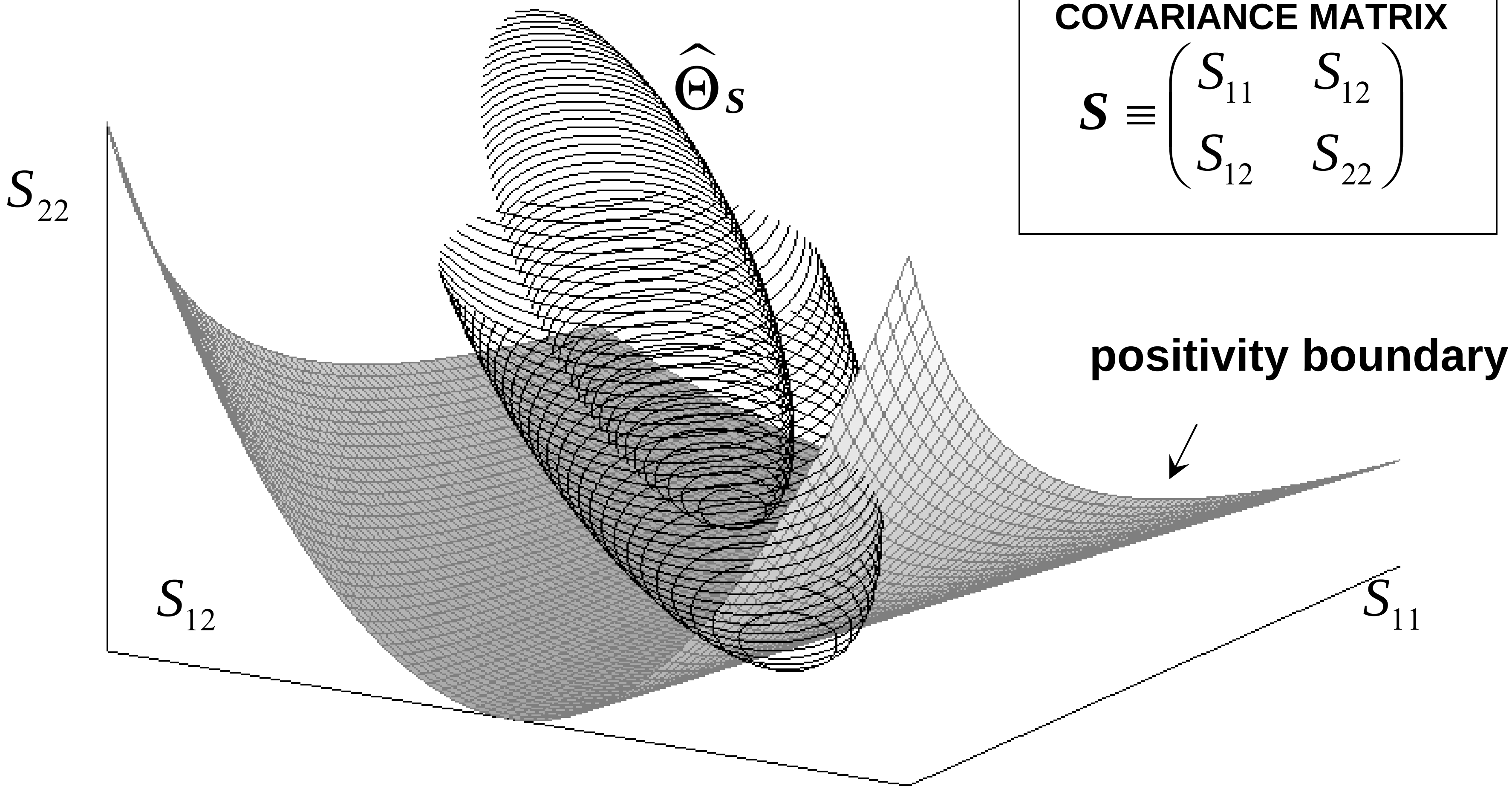
$\widehat{m}$: (point) estimate of m

$\widehat{S}$: (point) estimate of S

$\widehat{\Theta}_m$: uncertainty set for m

$\widehat{\Theta}_S$: uncertainty set for S

ROBUST OPTIMIZATION – uncertainty regions



EXAMPLE: 2x2 COVARIANCE MATRIX

$$\boldsymbol{S} \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix}$$

Trade-off for the choice of the uncertainty regions:

- Must be as large as possible, in such a way that the true, unknown parameters (most likely) are captured
- Must be as small as possible, to avoid trivial and nonsensical results

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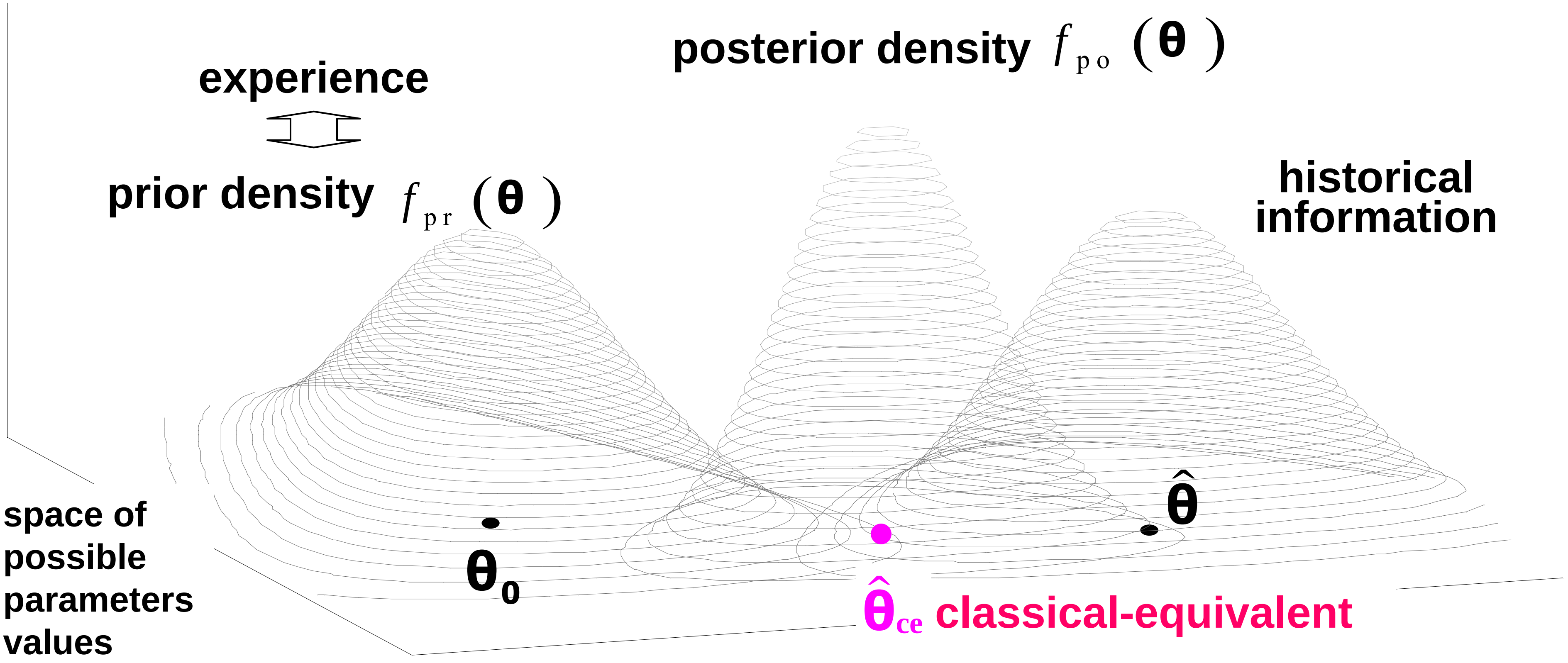
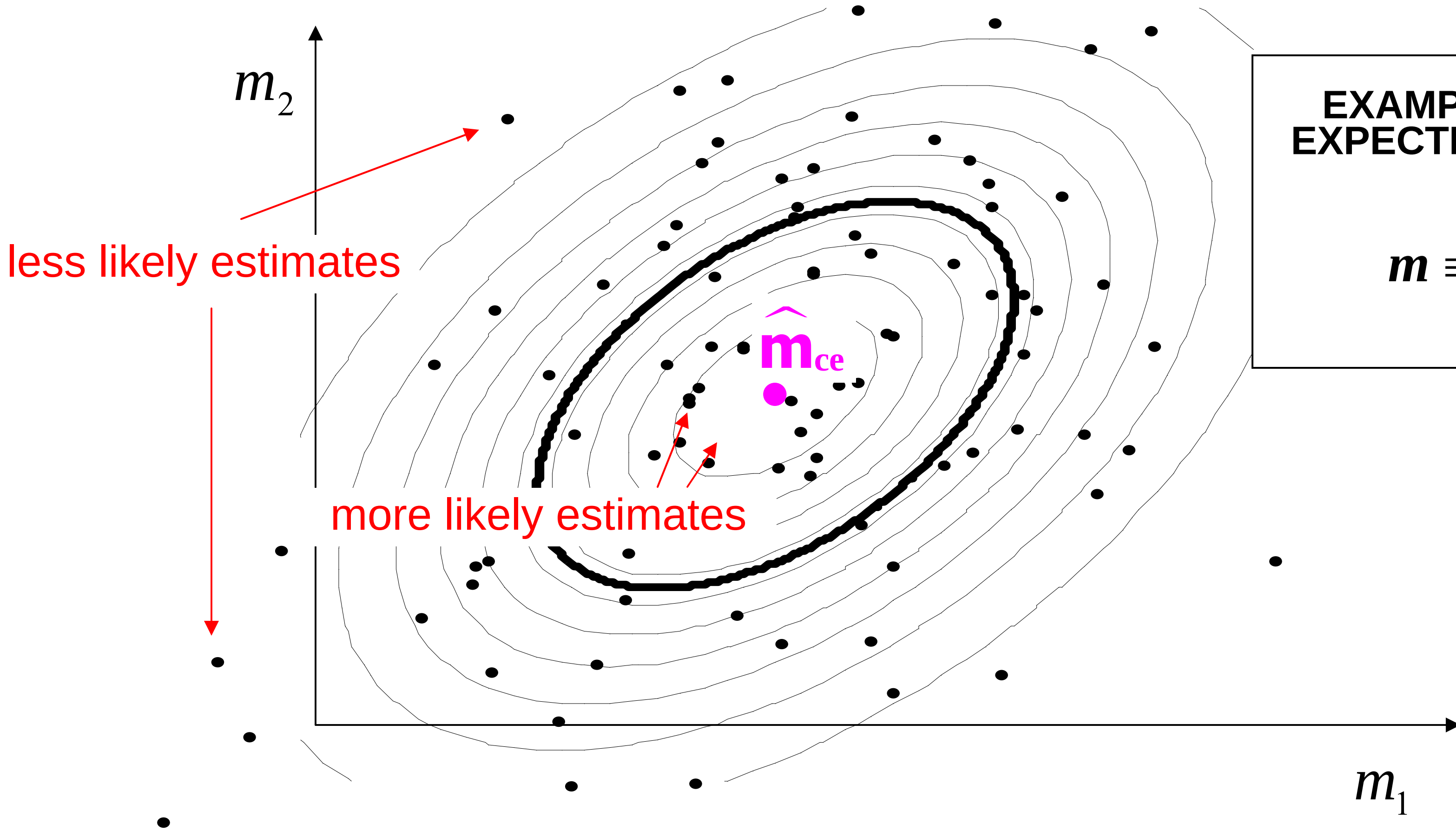
BAYESIAN OPTIMIZATION – Bayesian estimation theory

The Bayesian approach to estimation of the generic market parameters $\boldsymbol{\theta} \equiv (\boldsymbol{m}, \boldsymbol{S})$ differs from the classical approach in two respects:

- it blends historical information from time series analysis with experience
- the outcome of the estimation process is a (posterior) distribution, instead of a number



BAYESIAN OPTIMIZATION - Bayesian estimation theory

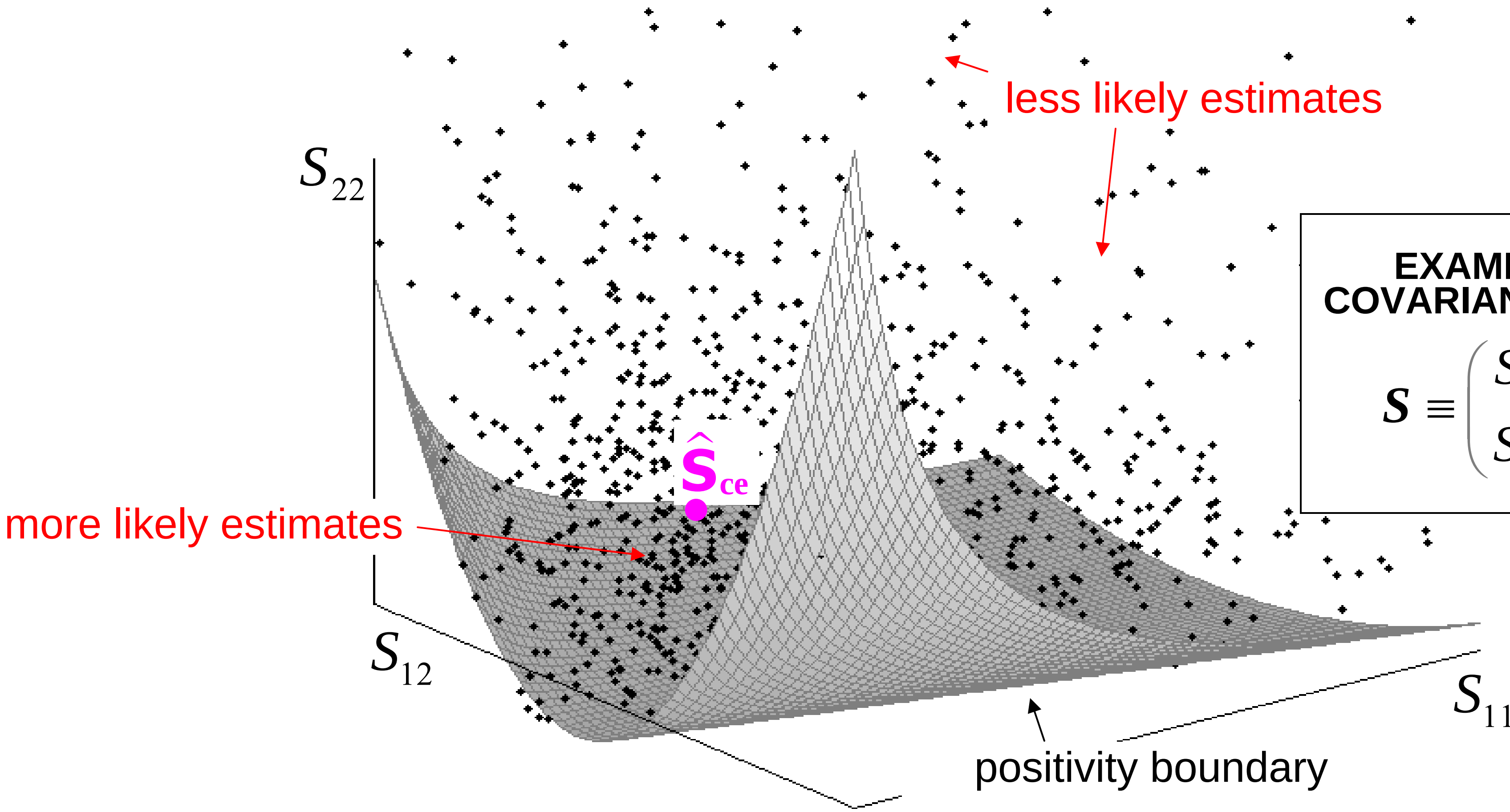


EXAMPLE: 2-dim EXPECTED VALUES

$$\boldsymbol{m} \equiv \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

in the Bayesian approach the expected values of the returns are a random variable

BAYESIAN OPTIMIZATION – Bayesian estimation theory



EXAMPLE: 2x2 COVARIANCE MATRIX

$$\boldsymbol{S} \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix}$$

in the Bayesian approach the covariance matrix of the returns is a random variable

BAYESIAN OPTIMIZATION – from the standard mean-variance …

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}}{\operatorname{argmax}} \left\{ \boldsymbol{w}' \widehat{\boldsymbol{m}} \right\}$$

$$\text{subject to} \begin{cases} \boldsymbol{w} \in \mathcal{C} \\ \boldsymbol{w}' \widehat{\boldsymbol{S}} \boldsymbol{w} \leq v^{(i)} \end{cases}$$

$\boldsymbol{w}$: relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $\boldsymbol{w}'\boldsymbol{1} = 1, \ \boldsymbol{w} \geq \boldsymbol{0}$

$v^{(i)}$: significant grid of target variances

$\widehat{\boldsymbol{m}}$: classical estimate of $\boldsymbol{m}$

$\widehat{\boldsymbol{S}}$: classical estimate of $\boldsymbol{S}$

BAYESIAN OPTIMIZATION – … to the classical-equivalent mean-variance

$$w^{(i)} \equiv \underset{w}{\operatorname{argmax}} \left\{ w' \widehat{m} \right\} \quad \text{subject to} \begin{cases} w \in \mathcal{C} \\ w' \widehat{S} w \leq v^{(i)} \end{cases}$$

$$w^{(i)} \equiv \underset{w}{\operatorname{argmax}} \left\{ w' \widehat{m}_{\mathbf{ce}} \right\} \quad \text{subject to} \begin{cases} w \in \mathcal{C} \\ w' \widehat{S}_{\mathbf{ce}} w \leq v^{(i)} \end{cases}$$

w : relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $w' \mathbf{1} = 1, \ w \geq \mathbf{0}$

$v^{(i)}$: significant grid of target variances

$\widehat{m}$: classical estimate of m

$\widehat{S}$: classical estimate of S

$\widehat{m}_{\mathbf{ce}}$: Bayesian classical-equivalent estimate for m

$\widehat{S}_{\mathbf{ce}}$: Bayesian classical-equivalent estimate for S

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ROBUST BAYESIAN OPTIMIZATION – the general framework

Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges

Bayesian allocations include the practitioner's experience

ROBUST BAYESIAN OPTIMIZATION – the general framework

Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges. However…

- the uncertainty regions for the market parameters are somewhat arbitrary
- the practitioner's experience, or prior knowledge, is not considered

Bayesian allocations include the practitioner's experience. However…

- the approach is not robust to estimation risk

Bayesian approach to parameter estimation within the robust framework

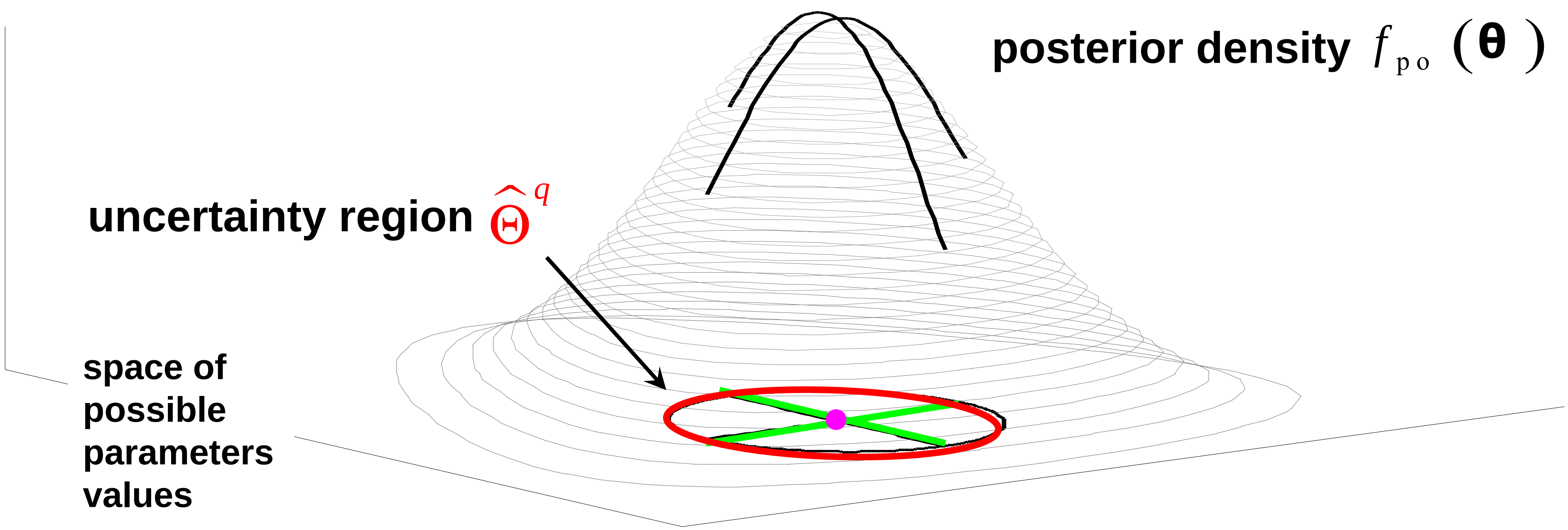
ROBUST BAYESIAN OPTIMIZATION – Bayesian ellipsoids

The Bayesian posterior distribution defines naturally a self-adjusting uncertainty region $\widehat{\Theta}^{q}$ for the market parameters

This region is the location-dispersion ellipsoid defined by

- a location parameter: the classical-equivalent estimator $\widehat{\boldsymbol{\theta}}_{ce}$
- a dispersion parameter: the positive symmetric scatter matrix $\mathbf{S}_{\boldsymbol{\theta}}$
- a radius factor q

$$\widehat{\Theta}^{q}: \quad \left(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}_{ce}\right)' \mathbf{S}_{\boldsymbol{\theta}}^{-1} \left(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}_{ce}\right) \leq q^{2}$$



ROBUST BAYESIAN OPTIMIZATION – Bayesian ellipsoids

Standard choices for the classical equivalent and the scatter matrix respectively:

- global picture: expected value / covariance matrix

$$\hat{\boldsymbol{\theta}}_{ce} \equiv \int \boldsymbol{\theta} f_{po}(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

$$\mathbf{S}_{\boldsymbol{\theta}} \equiv \int \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{ce}\right)\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{ce}\right)' f_{po}(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

- local picture: mode / modal dispersion

$$\hat{\boldsymbol{\theta}}_{ce} \equiv \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \left\{ f_{po}(\boldsymbol{\theta}) \right\}$$

$$\mathbf{S}_{\boldsymbol{\theta}} \equiv -\left(\left. \frac{\partial \ln f_{po}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right|_{\hat{\boldsymbol{\theta}}_{ce}} \right)^{-1}$$

ROBUST BAYESIAN OPTIMIZATION – from the standard mean-variance …

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\widehat{\boldsymbol{m}} \right\}$$

subject to $\boldsymbol{w}'\widehat{\boldsymbol{S}}\boldsymbol{w} \leq v^{(i)}$

$\boldsymbol{w}$: relative portfolio weights

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$v^{(i)}$: significant grid of target variances

$\widehat{\boldsymbol{m}}$: (point) estimate of $\boldsymbol{m}$

$\widehat{\boldsymbol{S}}$: (point) estimate of $\boldsymbol{S}$

ROBUST BAYESIAN OPTIMIZATION – … to the robust mean-variance …

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\widehat{\boldsymbol{m}} \right\}$$

subject to $\boldsymbol{w}'\widehat{\boldsymbol{S}}\boldsymbol{w} \leq v^{(i)}$

$$\boldsymbol{w}^{(i)} \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \min_{\boldsymbol{m}\in\widehat{\Theta}_{\boldsymbol{m}}} \boldsymbol{w}'\boldsymbol{m} \right\}$$

subject to $\max_{\boldsymbol{S}\in\widehat{\Theta}_{\boldsymbol{S}}} \left\{ \boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \right\} \leq v^{(i)}$

$\boldsymbol{w}$: relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $\boldsymbol{w}'\boldsymbol{1} = 1,\ \boldsymbol{w} \geq \boldsymbol{0}$

$v^{(i)}$: significant grid of target variances

$\widehat{\boldsymbol{m}}$: (point) estimate of $\boldsymbol{m}$

$\widehat{\boldsymbol{S}}$: (point) estimate of $\boldsymbol{S}$

$\widehat{\Theta}_{\boldsymbol{m}}$: uncertainty set for $\boldsymbol{m}$

$\widehat{\Theta}_{\boldsymbol{S}}$: uncertainty set for $\boldsymbol{S}$

ROBUST BAYESIAN OPTIMIZATION – … to the robust Bayesian MV

$$w^{(i)} \equiv \underset{w \in \mathcal{C}}{\operatorname{argmax}} \left\{ w' \widehat{m} \right\}$$

subject to $w' \widehat{S} w \leq v^{(i)}$

$$w_{p,q}^{(i)} \equiv \underset{w \in \mathcal{C}}{\operatorname{argmax}} \left\{ \min_{m \in \widehat{\Theta}_m^q} w' m \right\}$$

subject to $\max_{S \in \widehat{\Theta}_S^p} \left\{ w' S w \right\} \leq v^{(i)}$

w : relative portfolio weights

$\mathcal{C}$: set of investment constraints, e.g. $w' 1 = 1, \ w \geq 0$

$v^{(i)}$: significant grid of target variances

$\widehat{m}$: (point) estimate of m

$\widehat{S}$: (point) estimate of S

$\widehat{\Theta}_m^q$: Bayesian ellipsoid of radius q for m

$\widehat{\Theta}_S^p$: Bayesian ellipsoid of radius p for S

ROBUST BAYESIAN OPTIMIZATION – 3-dim. mean-variance frontier

The robust Bayesian efficient allocations $\mathbf{w}_{p,q}^{(i)}$ represent a **three-dimensional frontier** parametrized by:

1. Exposure to market risk represented by the target variance $\mathbf{v}^{(i)}$

2. Aversion to estimation risk for the expected returns $\mathbf{m}$ represented by radius $\mathbf{q}$

...indeed, a large ellipsoid $\widehat{\Theta}_{\mathbf{m}}^{q}$ corresponds to an investor that is very worried about poor estimates of $\mathbf{m}$

3. Aversion to estimation risk for the returns covariance $\mathbf{S}$ represented by radius $\mathbf{p}$

…indeed, a large ellipsoid $\widehat{\Theta}_{\mathbf{S}}^{p}$ corresponds to an investor that is very worried about poor estimates of $\mathbf{S}$

RBO EXAMPLE – market model

We make the following assumptions:

• The market is composed of equity-like securities, for which the returns are independent and identically distributed across time

• The estimation interval coincides with the investment horizon

• The linear returns are normally distributed:

$$\boldsymbol{L}_{t+\tau}^{\tau} \mid \boldsymbol{m}, \boldsymbol{S} \sim \mathrm{N}(\boldsymbol{m}, \boldsymbol{S})$$

We model the investor's prior as a normal-inverse-Wishart distribution:

$$\boldsymbol{m} \mid \boldsymbol{S} \sim \mathrm{N}\left(\boldsymbol{m}_0, \frac{\boldsymbol{S}}{T_0}\right), \quad \boldsymbol{S}^{-1} \sim \mathrm{W}\left(\nu_0, \frac{\boldsymbol{S}_0^{-1}}{\nu_0}\right)$$

where

$(\boldsymbol{m}_0, \boldsymbol{S}_0)$: investor's experience on $(\boldsymbol{m}, \boldsymbol{S})$

(T_0, ν_0) : investor's confidence on $(\boldsymbol{m}_0, \boldsymbol{S}_0)$

RBO EXAMPLE – posterior distribution of market parameters

Under the above assumptions, the posterior distribution is normal-inverse-Wishart, see e.g. Aitchison and Dunsmore (1975):

$$\boldsymbol{m} \mid \boldsymbol{S} \sim \mathrm{N}\left(\boldsymbol{m}_1, \frac{\boldsymbol{S}}{T_1}\right), \qquad \boldsymbol{S}^{-1} \sim \mathrm{W}\left(\nu_1, \frac{\boldsymbol{S}_1^{-1}}{\nu_1}\right)$$

where

$$\widehat{\boldsymbol{m}} \equiv \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{l}_t^{\tau} \qquad \widehat{\boldsymbol{S}} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{l}_t^{\tau} - \widehat{\boldsymbol{m}}\right)\left(\boldsymbol{l}_t^{\tau} - \widehat{\boldsymbol{m}}\right)'$$

$$T_1 \equiv T_0 + T \qquad \nu_1 \equiv \nu_0 + T$$

$$\widehat{\boldsymbol{m}}_1 \equiv \frac{1}{T_1}\left[T_0 \boldsymbol{m}_0 + T\widehat{\boldsymbol{m}}\right] \qquad \widehat{\boldsymbol{S}}_1 \equiv \frac{1}{\nu_1}\left[\nu_0 \boldsymbol{S}_0 + T\widehat{\boldsymbol{S}} + \frac{\left(\boldsymbol{m}_0 - \widehat{\boldsymbol{m}}\right)\left(\boldsymbol{m}_0 - \widehat{\boldsymbol{m}}\right)'}{\frac{1}{T_0} + \frac{1}{T}}\right]$$

RBO EXAMPLE – location-dispersion ellipsoids in practice

The certainty equivalent and the scatter matrix for the posterior (Student t) marginal distribution of $\boldsymbol{m}$ are computed in Meucci (2005):

$$\boldsymbol{m}_{\text{ce}} = \boldsymbol{m}_1, \qquad \boldsymbol{S}_{\boldsymbol{m}} = \frac{1}{T_1} \frac{\nu_1}{\nu_1 - 2} \boldsymbol{S}_1$$

The certainty equivalent and the scatter matrix for the posterior (inverse-Wishart) marginal distribution of $\boldsymbol{S}$ are computed in Meucci (2005):

$$\boldsymbol{S}_{\text{ce}} = \frac{\nu_1}{\nu_1 + N + 1} \boldsymbol{S}_1, \qquad \boldsymbol{S}_{\boldsymbol{S}} = \frac{2\nu_1^2}{(\nu_1 + N + 1)^3} \left(\mathbf{D}_{\text{N}}' \left(\boldsymbol{S}_1^{-1} \otimes \boldsymbol{S}_1^{-1} \right) \mathbf{D}_{\text{N}} \right)^{-1}$$

where $\mathbf{D}_{\text{N}}$ is the duplication matrix (see Magnus and Neudecker, 1999) and $\otimes$ is the Kronecker product

RBO EXAMPLE – efficient frontier

Under the above assumptions the robust Bayesian mean-variance problem:

$$\boldsymbol{w}_{p,q}^{(i)} \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \min_{\boldsymbol{m}\in\widehat{\Theta}_{\boldsymbol{m}}^{q}} \boldsymbol{w}'\boldsymbol{m} \right\}$$

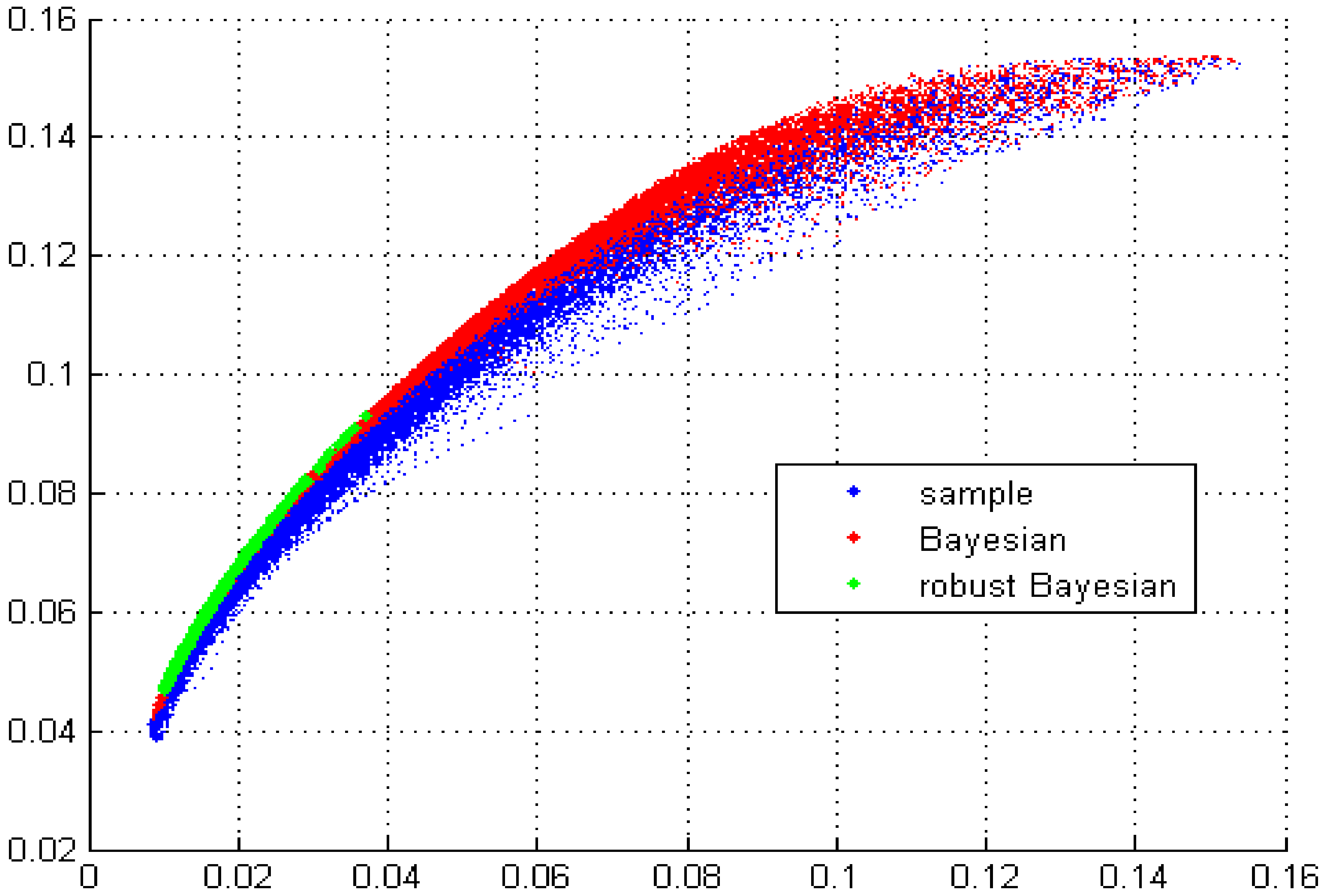
$$\text{subject to} \quad \max_{\boldsymbol{S}\in\widehat{\Theta}_{\boldsymbol{S}}^{p}} \left\{ \boldsymbol{w}'\boldsymbol{S}\boldsymbol{w} \right\} \leq v^{(i)}$$

…simplifies as follows:

$$\boldsymbol{w}_{p,q}^{(i)} \subset \boldsymbol{w}(\lambda) \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{m}_1 - \lambda\sqrt{\boldsymbol{w}'\boldsymbol{S}_1\boldsymbol{w}} \right\}$$

- The **three-dimensional frontier** collapses to a **line**
- The efficient frontier is parametrized by the exposure to **overall risk**, which includes market risk, estimation risk for $\boldsymbol{m}$ and estimation risk for $\boldsymbol{S}$

RBO EXAMPLE – efficient frontier



RBO EXAMPLE – robust Bayesian self-adjusting nature

• When the number of historical observations is large the uncertainty regions collapse to classical sample point estimates:

$$\boldsymbol{w}(\lambda) \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\widehat{\boldsymbol{m}} - \lambda\sqrt{\boldsymbol{w}'\widehat{\boldsymbol{S}}\boldsymbol{w}} \right\}$$

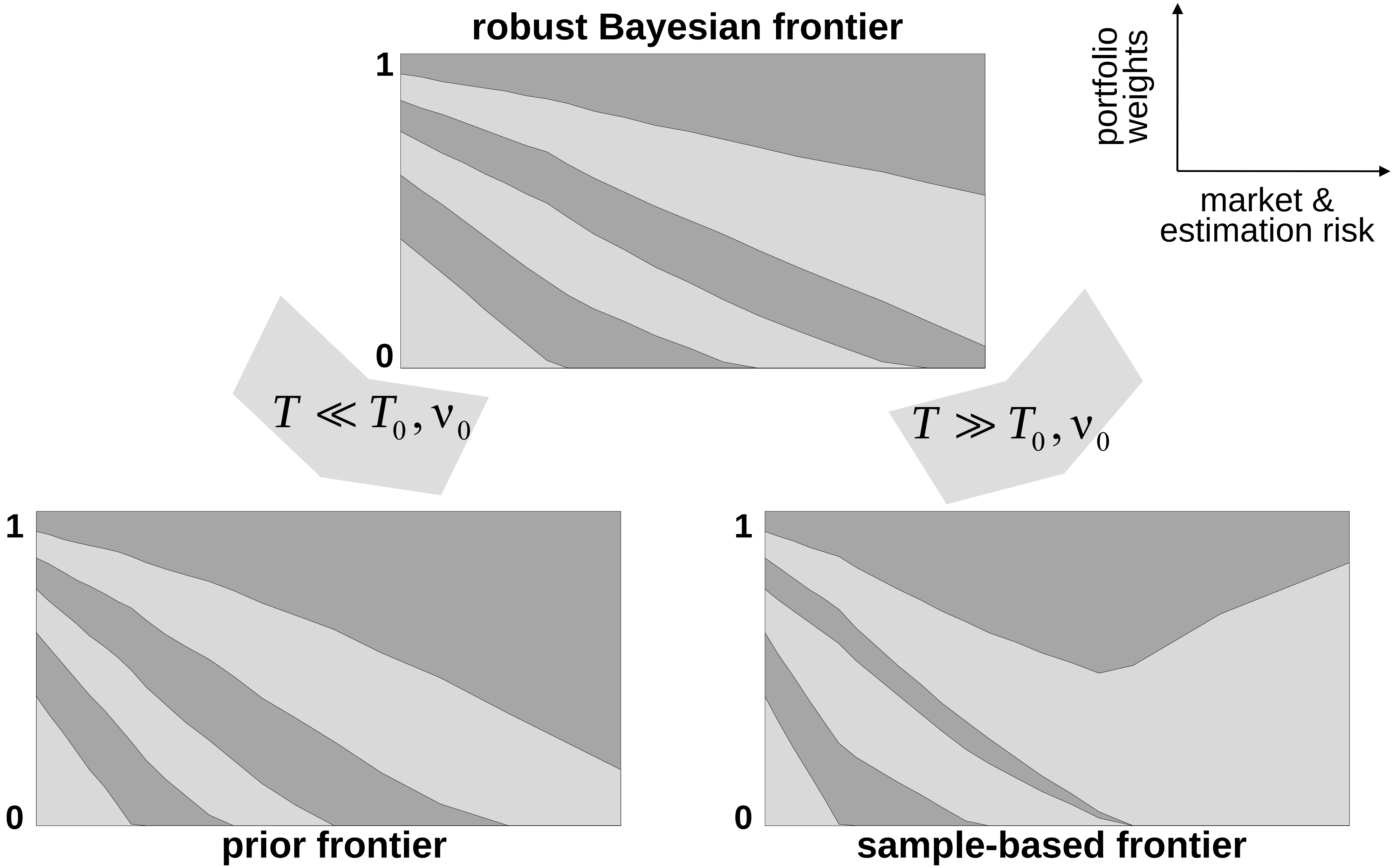
robust Bayesian frontier = classical sample-based frontier

• When the confidence in the prior is large the uncertainty regions collapse to the prior parameters:

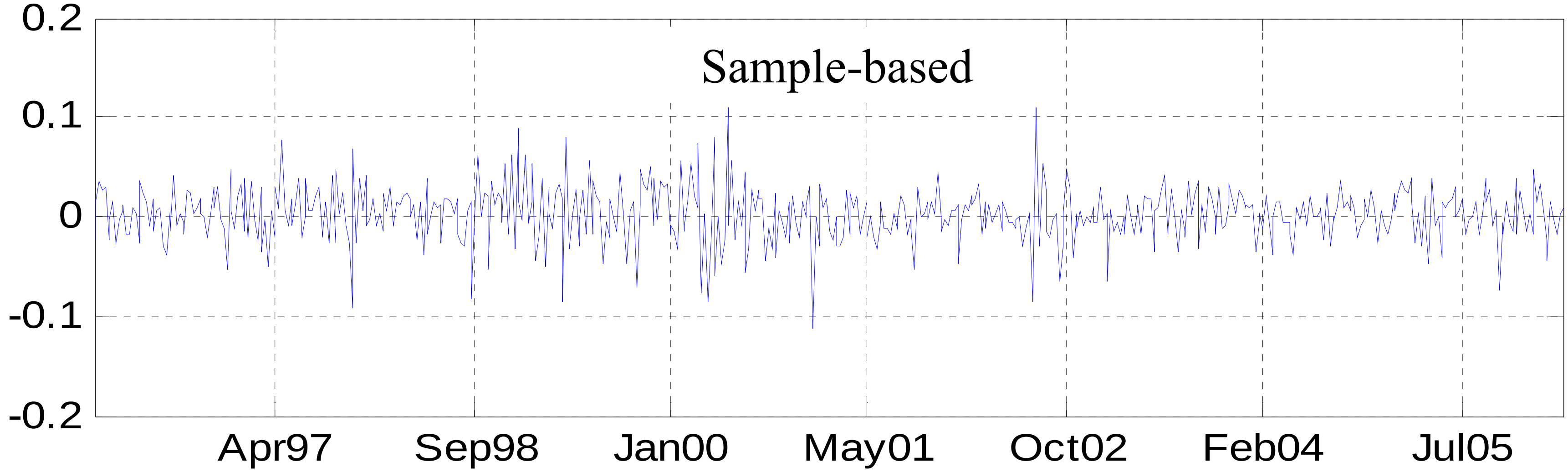
$$\boldsymbol{w}(\lambda) \equiv \underset{\boldsymbol{w}\in\mathcal{C}}{\operatorname{argmax}} \left\{ \boldsymbol{w}'\boldsymbol{m}_0 - \lambda\sqrt{\boldsymbol{w}'\boldsymbol{S}_0\boldsymbol{w}} \right\}$$

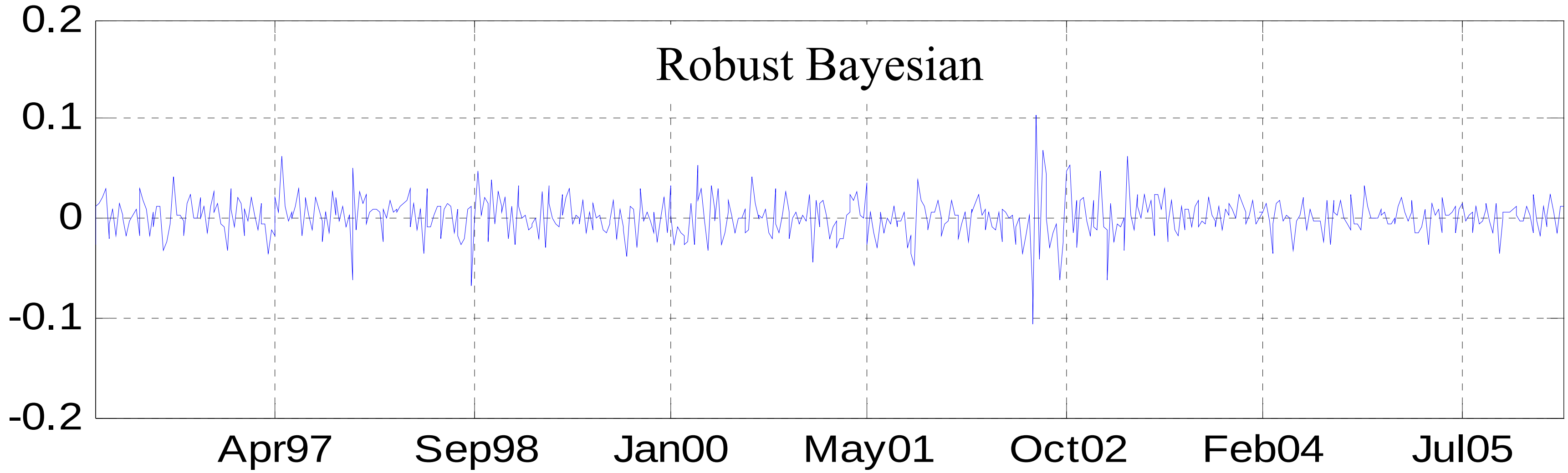
robust Bayesian frontier = “a-priori” frontier (no information from the market)

RBO EXAMPLE – robust Bayesian self-adjusting nature

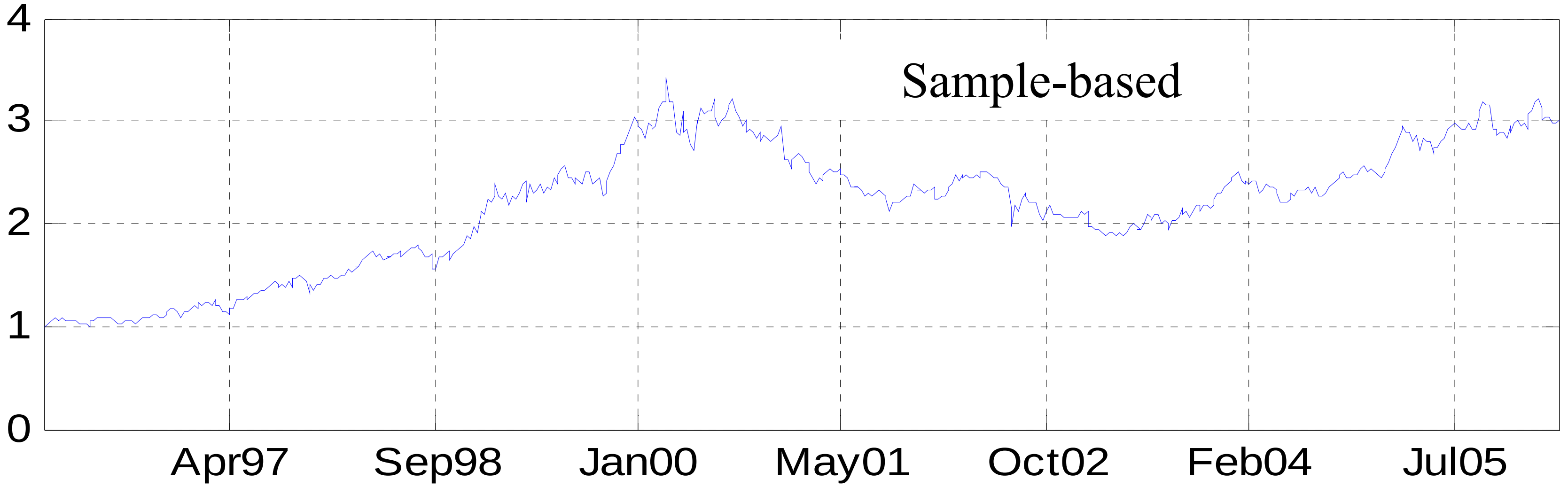


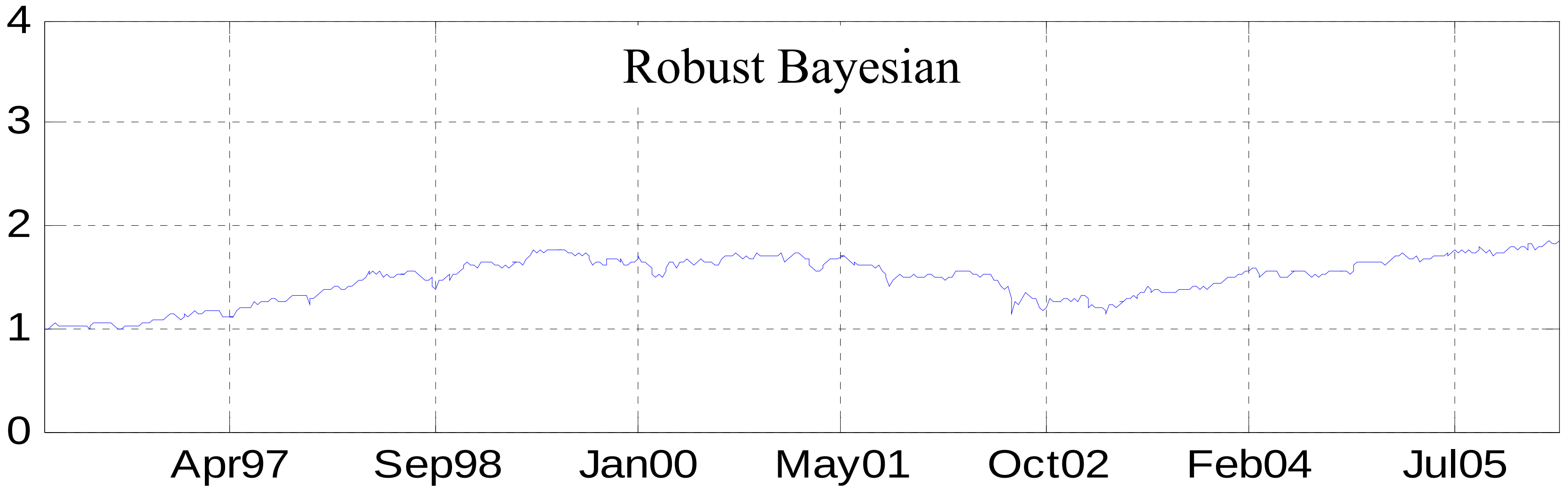
RBO EXAMPLE – robust Bayesian conservative nature (S&P 500)





RBO EXAMPLE – robust Bayesian conservative nature (S&P 500)





AGENDA

Estimation vs. Modeling

Classical Optimization and Estimation Risk

Black-Litterman Optimization

Robust Optimization

Bayesian Optimization

Robust Bayesian Optimization

References

REFERENCES

- This presentation:
  symmys.com > Teaching > Talks > *Issues in Quantitative Portfolio Management: Handling Estimation Risk*

- implementation code (MATLAB):
  symmys.com > Book > Downloads > MATLAB

- Comprehensive discussion of

  - modeling
  - estimation
  - location-dispersion ellipsoid
  - satisfaction maximization
  - quantitative portfolio-management
  - risk-management
  - estimation risk
  - Black-Litterman allocation
  - Bayesian techniques
  - robust techniques
  - …

  symmys.com > Book > A. Meucci, *Risk and Asset Allocation* - Springer (2005)