Optimal Portfolio Re-balancing

John A. Dodson*

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1 Problem Statement

We wish to formulate a version of the portfolio re-balancing problem in terms of linear programming. Our goal is to minimize transactions costs and capital gains taxes while returning the holdings weights to near their original values.

2 Decision Variables

The problem above is written in terms of the portfolio weights, but in fact we are interested in determining the optimal set of transactions. To that end, we will define the decicion variables as the quantity of each asset to buy or sell. Let us denote this by

$$x = (b_1, s_1, b_2, s_2, \ldots)' \tag{1}$$

which is a vector of non-negative numbers twice as long as the number of assets in the portfoio.

3 Data

Assume we know the quantity and price of each asset at the beginning of the period and also the current prices. Let us denote these quantites by α_i for the initial allocation (in shares), and p_i, and P_i for the initial and current prices. All of these are non-negative.

4 Parameters

4.1 Transactions Costs

We need to quantify the transactions costs and the capital gains taxes. Let us say denote the transaction costs (fees, duties, and bid-ask spread) by ϵ per share bought or sold. Therefore, the prices above should reflect fair mid-market valuations.

*University of Minnesota Financial Mathematics program & RiverSource Investments investment risk management

4.2 Capital Gains Taxes

Let us assume that taxes are incurred on every sale. If the sale price less transactions costs are higher than the initial price, then a tax liability is generated. Otherwise, a negative tax liability is generated. We will denote the tax rate by δ.

Note that we are assuming that we can carry over capital gains losses and that the initial price is equal to the tax basis for each asset.

4.3 Weight

We will assume that the initial weights were based on an asset allocation model, and remain optimal–that is why it is important to re-balance the portfolio. Since the weights will immediately drift again once the trade is complete, it is reasonable to allow some latitude in this process. Let us quantify this allowed slackness by a η and stipulate that each portfolio weight immediately after re-balancing should not differ from the initial weight by more than this amount.

5 Objective

The objective is to minimize costs. According to the definitions above, the cost associated with the trade list x is

$$\sum_i \epsilon \cdot (b_i + s_i) + \delta \cdot s_i \cdot (P_i - p_i - \epsilon) \tag{2}$$

Expressing this in the form $f' \cdot x$, we can get that

$$f = (\epsilon, \epsilon + \delta \cdot (P_1 - p_1 - \epsilon), \ldots)' \tag{3}$$

6 Constraints

6.1 Weights

The portfolio weights are

$$w_i = \frac{\alpha_i \cdot p_i}{\sum_j \alpha_j \cdot p_j} \qquad W_i = \frac{(\alpha_i + b_i - s_i) \cdot P_i}{\sum_j (\alpha_j + b_j - s_j) \cdot P_j} \tag{4}$$

for each asset i, ignoring uninvested cash and tax obligations.

In order to ensure that the new weights are close enough, we need $|W_i - w_i| \leq \eta$ for each asset i. Removing the absolute values, this is equivalent to

$$W_i - w_i \leq \eta$$
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W_i is not linear in the decision variables, but there are linear versions of these constraints. For example, $W_i - w_i \leq \eta$ is equivalent to

$$(b_i - s_i) \cdot P_i - (w_i + \eta) \cdot \sum_j (b_j - s_j) \cdot P_j \leq -\alpha_i \cdot P_i + (w_i + \eta) \cdot V \tag{5}$$

with $V = \sum_j \alpha_j \cdot P_j$ the total of the portfolio immediately before rebalancing.

6.2 Short-Sales

In order to forbid short sales, we must ensure that we do not sell more shares than we own. That is,

$$s_i \leq \alpha_i \tag{6}$$

6.3 Cash

Assets must be sold to cover purchases and costs. Therefore

$$f' \cdot x + \sum_i (b_i - s_i) \cdot P_i \leq 0 \tag{7}$$

7 Standard Form

The standard form for a linear programming problem is

$$\begin{aligned} &\min_x f' \cdot x \\ \text{s.t.} \quad &A \cdot x \leq b \\ &x \geq 0 \end{aligned} \tag{8}$$

Say there are N assets. x and f are both vectors of length $2 \cdot N$ which we have already discussed. We have enumerated a total of $3 \cdot N + 1$ linear constraints, so A and b have that many rows. These are of the form

$$\begin{aligned} (-(w_1 + \eta) \cdot P_1 + P_1, (w_1 + \eta) \cdot P_1 - P_1, \ldots) \cdot x &\leq -\alpha_1 \cdot P_1 + (w_1 + \eta) \cdot V \\ (-(w_2 + \eta) \cdot P_1, (w_2 + \eta) \cdot P_1, \ldots) \cdot x &\leq -\alpha_2 \cdot P_2 + (w_2 + \eta) \cdot V \\ &\vdots \\ ((w_1 - \eta) \cdot P_1 - P_1, -(w_1 - \eta) \cdot P_1 + P_1, \ldots) \cdot x &\leq \alpha_1 \cdot P_1 - (w_1 - \eta) \cdot V \\ ((w_2 - \eta) \cdot P_1, -(w_2 - \eta) \cdot P_1, \ldots) \cdot x &\leq \alpha_2 \cdot P_2 - (w_2 - \eta) \cdot V \\ &\vdots \end{aligned}$$

for the weights constraints;

$$\begin{aligned} (0, 1, 0, 0, \ldots) \cdot x &\leq \alpha_1 \\ (0, 0, 0, 1, \ldots) \cdot x &\leq \alpha_2 \\ &\vdots \end{aligned}$$

for the short-sale constraints; and

$$(f_1 + P_1, f_2 - P_1, f_3 + P_2, f_4 - P_2, \ldots) \cdot x \leq 0$$

for the cash constraint.

7.1 Intra-period Transactions

We need to consider one more constraint, which unfortunately cannot be linearized. If we allow an asset to be bought and sold in the same period, the cost basis per share for tax purposes is not just the inital price p_i, but rather some combination involving the current price (a a proxy for the actual intra-period transaction price).

$$p_i \rightarrow p_i + \frac{b_i}{\alpha_i + b_i} \cdot (P_i - p_i) \tag{9}$$

Clearly, this cannot be represented in the form $f' \cdot x$.

One approach to handle this is to forbid buying and selling the same asset in the same period. Since we can assume that s_i and b_i are both non-negative, this constraint can be compactly represented as

$$\sum_i b_i \cdot s_i = 0 \tag{10}$$

We can express this as a quadratic constraint, or instead as a penalty in the objective function.

$$\begin{aligned} \min_x f' \cdot x + \lambda \cdot \tfrac{1}{2} \cdot x' \cdot H \cdot x \\ \text{s.t.} \quad A \cdot x \leq b \\ x \geq 0 \end{aligned} \tag{11}$$

for some suitable $\lambda > 0$, where H is a symmetric matrix representing the LHS of (10).

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & \\ \vdots & \vdots & & & \ddots \end{pmatrix} \tag{12}$$

8 Example

The file –

`http://www.math.umn.edu/~dodso013/fm503/docs/case9.dat`

contains data for ten assets. It we assign $\epsilon = 0.05$, $\delta = 0.20$, and $\eta = 0.01$ for the transaction cost per share, capital gains tax rate, and the weight deviation range, we get the values for f, A, and b in the file –

http://www.math.umn.edu/~dodso013/fm503/docs/case9.zip

Running the quadratic optimization with the H from above, we get that the optimum re-balance trade is

asset 1	buy 5.5625
asset 3	buy 5.0314
asset 5	sell 1.2832
asset 7	sell 0.3964
asset 9	sell 2.3213
asset 10	sell 5.6299

which generates costs of 3.2417, or about 0.015 % of the portfolio value. This consists of about 1.01 in transaction costs on 20.2 shares, and about 2.23 in net capital gains taxes on the four sales. The capital gains taxes are signifigantly reduced by selling asset 10 at a loss while maintaining its weight at the bottom of the allowed range.

By way of comparison, a naive re-balancing procedure might involve matching exactly the intial weights without regard to costs. This would require transacting in more than twice as many shares, generating 2.07 in transactions costs and 22.49 in capital gains taxes. The optimal solution we found is almost 90 % cheaper (at least in the short term).