“Square-root Rule” for Time Scaling Market Invariants

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Say X is the random variable that will drive market prices between today, time T, and the next decision date, time $T+\tau$ with an investment horizon of τ (all time in years).

$$P_{T+\tau} = g(X; P_T)$$

and say that we have used historical data to estimate the parameters of the distribution of the market invariants under a different horizon, $\tilde{\tau}$; e.g., from $\{P_T, P_{T-\tilde{\tau}}, P_{T-2\cdot\tilde{\tau}}\dots,\}$.

We know from the properties of characteristic functions that

$$\phi_X(t) = \phi_Y(t)^{\frac{\tau}{\tilde{\tau}}}$$

If the first two moments exist, we also know that

$$\begin{aligned}
\mathsf{E}X &= -i \cdot \left.\frac{d\phi_X}{dt'}\right|_0 \\
&= -i \cdot \frac{\tau}{\tilde{\tau}} \cdot \phi_Y^{\frac{\tau}{\tilde{\tau}}-1} \cdot \left.\frac{d\phi_Y}{dt'}\right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \cdot \mathsf{E}Y
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{E}\left(X \cdot X'\right) &= -\left.\frac{d^2\phi_X}{dt'\,dt}\right|_0 \\
&= -\frac{\tau}{\tilde{\tau}} \cdot \left(\frac{\tau}{\tilde{\tau}} - 1\right) \cdot (\phi_Y)^{\frac{\tau}{\tilde{\tau}}-2} \cdot \frac{d\phi_Y}{dt'} \cdot \frac{d\phi_Y}{dt} - \frac{\tau}{\tilde{\tau}} \cdot (\phi_Y)^{\frac{\tau}{\tilde{\tau}}-1} \cdot \left.\frac{d^2\phi_Y}{dt'\,dt}\right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \cdot \left(\frac{\tau}{\tilde{\tau}} - 1\right) \cdot \mathsf{E}Y \cdot \mathsf{E}Y' + \frac{\tau}{\tilde{\tau}} \cdot \mathsf{E}\left(Y \cdot Y'\right) \\
&= \mathsf{E}X \cdot \mathsf{E}X' + \frac{\tau}{\tilde{\tau}} \cdot \left(\mathsf{E}\left(Y \cdot Y'\right) - \mathsf{E}Y \cdot \mathsf{E}Y'\right)
\end{aligned}$$

Since the covariance is defined as

$$\operatorname{cov} Y = \mathsf{E}\left(X \cdot X'\right) - \mathsf{E}X \cdot \mathsf{E}X'$$

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we have that

$$\operatorname{cov} X = \frac{\tau}{\tilde{\tau}} \cdot \operatorname{cov} Y$$

Furthermore, since

$$\operatorname{std} Y = \operatorname{diag} \sqrt{\operatorname{diag} \operatorname{diag} \operatorname{cov} Y}$$

we have the “square-root rule” for time-scaling market invariants.

$$\operatorname{std} X = \sqrt{\frac{\tau}{\tilde{\tau}}} \cdot \operatorname{std} Y$$

This is valid result regardless of the distribution of Y (as long as it has two moments). In general, X will not belong to the same class of distributions as Y, unless Y is in the stable family, including the normal, Cauchy, and Lévy.