

Risk & Asset Allocation (Spring)

Case for Week 3

John A. Dodson

February 6, 2013

Let us consider the expected shortfall index of satisfaction for a very simple portfolio: α shares in an asset whose value today is $p > 0$ and whose horizon value P is lognormal.

Let us assume that the objective measure is profit; therefore in Meucci's notation, we have

$$\begin{aligned}\Psi_\alpha &= \alpha M \\ &= \alpha (P - p) \\ &= \alpha (g(X) - p) \\ &= \alpha p (e^X - 1)\end{aligned}$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean μ and variance $\Sigma > 0$. The index of satisfaction is

$$\mathcal{S}(\alpha) = \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_\alpha}(q) dq$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$\begin{aligned}F_{\Psi_\alpha}(z) &= \mathbb{P}\{\Psi_\alpha < z\} \\ &= \mathbb{P}\{\alpha p (e^X - 1) < z\} \\ &= \mathbb{P}\left\{X \operatorname{sgn} \alpha < \log\left(1 + \frac{z}{\alpha p}\right) \operatorname{sgn} \alpha\right\} \\ &= \mathbb{P}\left\{\frac{X - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \alpha < \frac{\log\left(1 + \frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \alpha\right\} \\ &= \Phi\left(\frac{\log\left(1 + \frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma} \operatorname{sgn} \alpha}\right)\end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the CDF, is therefore

$$Q_{\Psi_\alpha}(q) = \alpha p \left(e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$\begin{aligned} \mathcal{S}(\alpha) &= \frac{1}{1-c} \int_0^{1-c} \alpha p \left(e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq \\ &= \alpha p \left(\frac{1}{1-c} \int_0^{1-c} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right) \\ &= \alpha p \left(\frac{1}{1-c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} z} \phi(z) dz - 1 \right) \end{aligned}$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} z} \phi(z) = e^{\mu + \frac{1}{2}\Sigma} \phi \left(z - \text{sgn } \alpha \sqrt{\Sigma} \right)$$

we have the final result,

$$\mathcal{S}(\alpha) = \alpha p \left(e^{\mu + \frac{1}{2}\Sigma} \frac{1}{1-c} \Phi \left(\Phi^{-1}(1-c) - \text{sgn } \alpha \sqrt{\Sigma} \right) - 1 \right) \quad (1)$$

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$\mathcal{S}(\alpha) \approx \alpha p \left(\mu - \text{sgn } \alpha \frac{\phi \left(\Phi^{-1}(1-c) \right)}{1-c} \sqrt{\Sigma} \right) \quad (2)$$

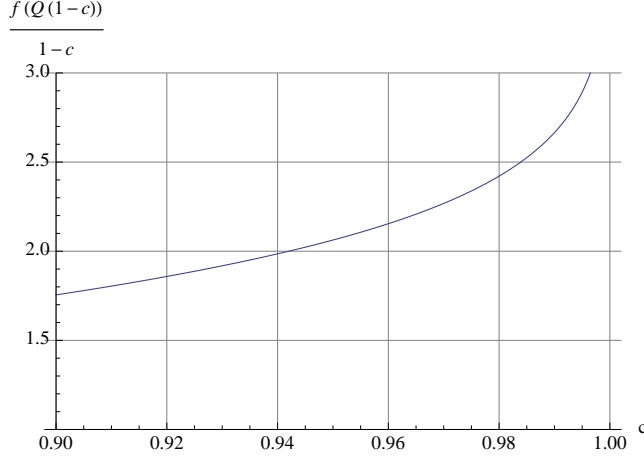
Let us spend a moment interpreting this. An investor will be more satisfied to be long ($\alpha > 0$) if the asset has a positive expected return ($\mu > 0$), and short ($\alpha < 0$) if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance diminishes satisfaction for any non-zero position.

This all seems quite reasonable for a rational index of satisfaction.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the objective Ψ_α . In a Delta-Gamma setting, we can replace the objective by the quadratic

$$\Psi_\alpha = \alpha p \left(e^X - 1 \right) \approx \alpha p \left(X + \frac{1}{2} X^2 \right)$$



hence $\Theta_\alpha = 0$, $\Delta_\alpha = \alpha p$, and $\Gamma_\alpha = \alpha p$. Let us define a new objective to represent this approximation.

$$\Xi_\alpha = \alpha p \left(X + \frac{1}{2} X^2 \right)$$

It is straight-forward to work out that the first several central moments of this are

$$\begin{aligned} E(\Xi_\alpha) &= \alpha p \left(\mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) \\ \text{Sd}(\Xi_\alpha) &= |\alpha| p \sqrt{\Sigma} \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} \\ \text{Sk}(\Xi_\alpha) &= 3 \operatorname{sgn} \alpha \sqrt{\Sigma} \frac{(1 + \mu)^2 + \frac{1}{3} \Sigma}{\left((1 + \mu)^2 + \frac{1}{2} \Sigma \right)^{3/2}} \end{aligned}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$\mathcal{S}(\alpha) \approx E(\Xi_\alpha) + \text{Sd}(\Xi_\alpha) \left(z_1 + \frac{z_2 - 1}{6} \text{Sk}(\Xi_\alpha) \right)$$

with coefficients

$$\begin{aligned} z_1 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q) dq = -\frac{\phi(\Phi^{-1}(1-c))}{1-c} \\ z_2 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)^2 dq = 1 - \frac{\phi(\Phi^{-1}(1-c))}{1-c} \Phi^{-1}(1-c) \end{aligned}$$

depending on the confidence level $c < 1$ ¹.

Putting this together, we get a third expression for the index of satisfaction.

$$\begin{aligned} \mathcal{S}(\alpha) &\approx \alpha p \left(\mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) - |\alpha| p \frac{\phi(\Phi^{-1}(1-c))}{1-c} \sqrt{\Sigma} \\ &\quad \cdot \left(\sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \operatorname{sgn} \alpha \frac{(1 + \mu)^2 + \frac{1}{3} \Sigma}{(1 + \mu)^2 + \frac{1}{2} \Sigma} \Phi^{-1}(1-c) \sqrt{\Sigma} \right) \quad (3) \end{aligned}$$

This result agrees with (2) to lowest order in μ and Σ .

¹The trick to these integrals is to realize that $\phi'(z) = -z\phi(z)$.

4 Modeling Default

Our horizon asset value P is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$\Psi'_\alpha = \alpha p (Y e^X - 1)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before², but now we add an independent default indicator $Y \sim \text{Bern}(1 - h)$ for default probability h .

²Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.