

# Risk & Asset Allocation (Spring)

## Exercise for Week 4

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Let us try out the two-step approach to determine the optimal portfolio under a single affine constraint with objective linear in the market vector and index of satisfaction equal to the Cornish-Fisher expansion of the 95% expected shortfall.

For an objective defined by  $\Psi_\alpha = \alpha' M$  and an affine constraint defined by  $d' \alpha = c$ , the analytic solution to the optimal mean-variance portfolio satisfies

$$\alpha(\beta) = (1 - \beta)\alpha_{MV} + \beta\alpha_{SR} \quad (1)$$

for  $\beta > 0$ , where

$$\alpha_{MV} = \frac{c (\text{Cov}M)^{-1} d}{d' (\text{Cov}M)^{-1} d}$$

$$\alpha_{SR} = \frac{c (\text{Cov}M)^{-1} EM}{d' (\text{Cov}M)^{-1} EM}$$

We need to determine the level of  $\beta$  that maximizes the index of satisfaction, which we will take to be

$$\mathcal{S}(\alpha) = E \Psi_\alpha + \sqrt{\text{Var} \Psi_\alpha} \left( \mathcal{I} [\phi \Phi^{-1}] + \frac{1}{6} \left( \mathcal{I} [\phi (\Phi^{-1})^2] - 1 \right) \text{Skew} \Psi_\alpha \right)$$

based on the Cornish-Fisher expansion, where  $\Phi$  is the CDF of a standard normal random variable and  $\phi$  is the spectrum for  $ES_{0.95}$ .

Since we can assume that the skewness of  $M$  is negligible, the skewness of  $\Psi_\alpha$  is also negligible. Furthermore, we can evaluate the integral in the expansion numerically.

$$\mathcal{I} [\phi \Phi^{-1}] = \int_0^{0.05} \frac{\sqrt{2} \text{erf}^{-1}(2p - 1)}{0.05} dp \approx -2.0627 \dots$$

Let us assign  $z_{0.95} = 2.0627 \dots$ , so the integral above is  $-z_{0.95}$ . The satisfaction is

$$\mathcal{S}(\alpha) = \alpha' EM - z_{0.95} \sqrt{\alpha' (\text{Cov}M) \alpha}$$

Substituting in (1), we get that the optimal value for  $\beta$  is

$$\beta^* = \arg \max_{\beta > 0} ((1 - \beta)\alpha_{MV} + \beta\alpha_{SR})' EM$$

$$- z_{0.95} \sqrt{((1 - \beta)\alpha_{MV} + \beta\alpha_{SR})' (\text{Cov}M) ((1 - \beta)\alpha_{MV} + \beta\alpha_{SR})}$$

From manipulation of the first-order condition, recognizing that

$$\text{Cov}(\Psi_{\alpha_{MV}}, \Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}) = 0$$

we can determine that the solution is

$$\beta^* = \begin{cases} 0 & \gamma \leq 0 \\ \sqrt{\frac{\text{Var} \Psi_{\alpha_{MV}}}{\text{Var}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}})} \frac{1}{\left(\frac{z_{0.95}}{\gamma}\right)^2 - 1}} & 0 < \gamma < z_{0.95} \\ \infty & \gamma \geq z_{0.95} \end{cases}$$

where

$$\gamma = \frac{\text{E}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}})}{\sqrt{\text{Var}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}})}}$$

is the market price for risk.

In conclusion, the optimal portfolio is  $\alpha(\beta^*)$ .