Risk & Asset Allocation (Spring)
Exercise for Week 6

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Consider a broad investment universe and a normal market vector. Assume all initial asset prices are one. Under the objective probability measure, each pair-wise correlation is $0 < \rho < 1$ and each component's mean and variance is μ_0 and σ_0^2 respectively.

The manager's subjective probability measure is based on a view with confidence $0 < c < 1$ that one particular market vector component will turn out to be $v_1 \in \mathbb{R}$.

The picks matrix is simply

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}$$

So $P\Sigma P'$ is just the scalar $[\Sigma]_{11} = \sigma_0^2$ and

$$\Sigma P' = \begin{pmatrix} \sigma_0^2 \\ \sigma_0^2 \rho \\ \sigma_0^2 \rho \\ \vdots \end{pmatrix}$$

The Black-Litterman market vector mean is

$$\mu_{BL} = \mu + c\Sigma P' \left(P\Sigma P'\right)^{-1} (v - P\mu) = \begin{pmatrix} (1-c)\mu_0 + cv_1 \\ (1-c\rho)\mu_0 + c\rho v_1 \\ (1-c\rho)\mu_0 + c\rho v_1 \\ \vdots \end{pmatrix} \tag{1}$$

Notice that the marginal variances factor out.

To evaluate the α_{SR} portfolio for the second question, we need first to evaluate

$$\Sigma_{BL} = \Sigma - c\Sigma P' \left(P\Sigma P'\right)^{-1} P\Sigma$$

Since Σ is symmetric, $\Sigma P' = (P\Sigma)'$. Thus we can arrive at

$$\Sigma_{BL} = \sigma_0^2 \begin{pmatrix} 1-c & \rho - c\rho & \rho - c\rho & \cdots \\ \rho - c\rho & 1 - c\rho^2 & \rho - c\rho^2 & \cdots \\ \rho - c\rho & \rho - c\rho^2 & 1 - c\rho^2 & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

The inverse of this is not necessarily apparent, but turns out to be

$$\Sigma_{BL}^{-1} = \frac{1}{\sigma_0^2(1-\rho)} \begin{pmatrix} \frac{1-\rho}{1-c} + \frac{(n-1)\rho^2}{1+(n-1)\rho} & -\frac{\rho}{1+(n-1)\rho} & -\frac{\rho}{1+(n-1)\rho} & \cdots \\ -\frac{\rho}{1+(n-1)\rho} & 1 - \frac{\rho}{1+(n-1)\rho} & -\frac{\rho}{1+(n-1)\rho} & \cdots \\ -\frac{\rho}{1+(n-1)\rho} & -\frac{\rho}{1+(n-1)\rho} & 1 - \frac{\rho}{1+(n-1)\rho} & \\ \vdots & \vdots & & \ddots \end{pmatrix} \tag{2}$$

where $n = \dim M$ is the number of assets in the investment universe. The proof of this comes from multiplying out Σ_{BL}^{-1} and Σ_{BL}.

Assuming the initial price vector p is one (or at least proportional to one), the SR portfolio allocation is proportional to

$$\alpha_{SR} \propto \Sigma_{BL}^{-1}\mu_{BL} = \frac{1}{\sigma_0^2} \begin{pmatrix} v_1 \frac{c}{1-c} + \frac{\mu_0}{1+(n-1)\rho} \\ \frac{\mu_0}{1+(n-1)\rho} \\ \frac{\mu_0}{1+(n-1)\rho} \\ \vdots \end{pmatrix}$$

The fraction of the initial value of this portfolio allocated to the first asset is

$$\frac{[p]_1 \, [\alpha_{SR}]_1}{p'\alpha_{SR}} = \frac{v_1 \frac{c}{1-c} + \frac{\mu_0}{1+(n-1)\rho}}{v_1 \frac{c}{1-c} + \frac{n\mu_0}{1+(n-1)\rho}}$$

For a sufficiently broad investment universe, this limits to

$$\lim_{n\to\infty} \frac{[p]_1 \, [\alpha_{SR}]_1}{p'\alpha_{SR}} = \frac{1}{1 + \frac{1-c}{c} \frac{\mu_0}{v_1} \frac{1}{\rho}} \tag{3}$$