Risk & Asset Allocation
Case for Week 6

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Implied Risk-Neutral Density

For a European call we have

$$c(t, T, K) = \mathrm{E}^{\mathbb{Q}} \left[D(t, T) \left(S_T - K \right)^+ \big| \mathcal{F}_t \right]$$

where D is the stochastic risk-free discount factor [2]. If innovations of the underlying are independent of innovations in the risk-free interest rate (or otherwise under the forward measure), we have

$$\frac{d^2 c}{dK^2} = \mathrm{E}^{\mathbb{Q}} \left[D(t, T) \delta \left(S_T - K \right) \big| \mathcal{F}_t \right]$$
$$= P(t, T) f_{S_T | \mathcal{F}_t}(K)$$

where $f_{S_T | \mathcal{F}_t}(\cdot)$ is the risk-neutral probability density of the terminal value of the underlying and $P(t, T)$ is the risk-free discount factor.

Since the risk-neutral expected value is just the forward value, we have

$$F(t, T) \triangleq \mathrm{E}^{\mathbb{Q}} \left[S_T | \mathcal{F}_t \right]$$
$$= \frac{1}{P(t, T)} \int_0^\infty K \frac{d^2 c}{dK^2} \, dK$$
$$= \frac{1}{P(t, T)} \left(K \frac{dc}{dK} - c \right) \Big|_{K=0}^\infty$$
$$= \frac{c|_{K=0}}{P(t, T)} = \frac{S_t - \int_t^T P(t, T') \, dv(T')}{P(t, T)}$$

where $v(\cdot)$ is the cumulative cashflow stream from the underlying asset. To verify the assertion, note that the net present value of the following stream is zero:

$$0 = \int_t^T P(t, T') \frac{d}{dT'} \left(-S_t \delta(T' - t) + v(T') + F(t, T) \delta(T' - T) \right) \, dT'$$

This is the definition of a forward, the fair price one could contract today to sell an asset at a future date.

European-style payoffs

In general, the risk-neutral expected value of any function of the terminal value of the underlying $h(\cdot)$ is

$$
\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left[h(S_T)\vert\,\mathcal{F}_t\right] &= \frac{1}{P(t,T)}\int_0^\infty h(K)\frac{d^2c}{dK^2}\,dK \\
&= h\left(F(t,T)\right) + \int_0^\infty h''(K)\left(\frac{c(t,T,K)}{P(t,T)} - (F(t,T)-K)^+\right)\,dK
\end{aligned}
$$

where the latter equality uses integration by parts (twice) to transfer the second derivative from the call strike structure to the payoff.

This provides a simple result for interpolating the value of any European-style derivative in terms of the strike structure of the time value of calls[1].

$$
\mathrm{E}^{\mathbb{Q}}\left[D(t,T)h(S_T)\vert\,\mathcal{F}_t\right] = P(t,T)h\left(F(t,T)\right) + \int_0^\infty h''(K)\left(c(t,T,K) - P(t,T)\left(F(t,T)-K\right)^+\right)\,dK
$$

It also gives us a means of analyzing the implied risk-neutral density of the future value of the underlying.

Implied variance

For example, with $h(S_T) = (S_T - F(t,T))^2$ we find that

$$
\mathrm{var}^{\mathbb{Q}}\left[S_T\vert\mathcal{F}_t\right] = 2\int_0^\infty \frac{c(t,T,K)}{P(t,T)}\,dK - F(t,T)^2 \tag{1}
$$

It is conventional to express European option premia in terms of implied volatility, σ_B, based on the Black formula.

$$
\begin{aligned}
c(t,T,K) = P(t,T)\Bigg(&F(t,T)\Phi\left(\frac{\log\frac{F(t,T)}{K}}{\sigma_B(t,T,K)\sqrt{\tau(t,T)}} + \tfrac{1}{2}\sigma_B(t,T,K)\sqrt{\tau(t,T)}\right) \\
&- K\Phi\left(\frac{\log\frac{F(t,T)}{K}}{\sigma_B(t,T,K)\sqrt{\tau(t,T)}} - \tfrac{1}{2}\sigma_B(t,T,K)\sqrt{\tau(t,T)}\right)\Bigg)
\end{aligned}
$$

where $\tau(t,T)$ is the option tenor expressed in years (e.g. business days divided by 252).

We will substitute this into (1) for the implied variance, but before we do so the form of the Black equation suggests a change of variables to $k = \log\frac{K}{F(t,T)}$, which is termed the option "moneyness".

$$
\mathrm{var}^{\mathbb{Q}}\left[S_T\vert\mathcal{F}_t\right] = F(t,T)^2\left(2\int_{-\infty}^\infty e^k\Phi\left(\frac{-k}{\varsigma(k)} + \tfrac{1}{2}\varsigma(k)\right) - e^{2k}\Phi\left(\frac{-k}{\varsigma(k)} - \tfrac{1}{2}\varsigma(k)\right)\,dk - 1\right) \tag{2}
$$

where

$$
\varsigma(k) \triangleq \sigma_B\left(t,T,F(t,T)e^k\right)\sqrt{\tau(t,T)}
$$

is the implied standard deviation of log returns expressed in terms of moneyness.

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[1]Cf. [1]

If the implied volatility is constant in strike, then the market conforms to the Back-Scholes log-normal model. In this case, the integral in (2) can be evaluated using integration by parts and a change of variables to yield the expected result,

$$\text{var}^{\mathbb{Q}}\left[S_T | \mathcal{F}_t\right] = F(t,T)^2 \left(e^{\sigma_B^2(t,T,\cdot)\tau(t,T)} - 1\right)$$

In general, though, the implied volatility is not the same for all strike prices. In this case, the integral in (2) may not be analytically tractable, but it is still numerically tractable.

SVI form

The results above depend on having options premia or implied volatility for a continuum of strike prices. In fitting actual options data, the SVI ("stochastic volatility inspired") functional form [3] can be useful.

$$\varsigma(k) \triangleq \sqrt{c_1 + c_2 k + \sqrt{c_3 + c_4 k + c_5 k^2}}$$

where presumably the five coefficients depend on the value and expiration dates t and T.

Implied Characteristic Function

It can be useful to characterize the log-return rather than the horizon value of an asset. Using $h(S_T) = \left(\frac{S_T}{F(t,T)}\right)^{iz}$ we can derive the characteristic function of the excess[2] log-return.

$$\phi^{\mathbb{Q}}_{\log \frac{S_T}{F(t,T)}|\mathcal{F}_t}(z) = 1 + \int_0^{\infty} \frac{iz(iz-1)}{K^2} \left(\frac{K}{F(t,T)}\right)^{iz} \left(\frac{c(t,T,K)}{P(t,T)} - (F(t,T)-K)^+\right) dK$$

$$= 1 - (z^2 + iz) \int_{-\infty}^{\infty} e^{izk} \left(\frac{c\left(t,T,F(t,T)e^k\right)}{P(t,T)F(t,T)e^k} - \left(e^{-k}-1\right)^+\right) dk$$

If we are interested in a different horizon, and we are willing at make the assumption that increments are additive, we can easily adjust the characteristic function

$$\phi^{\mathbb{Q}}_{\log \frac{S_{t+\tau}}{F(t,t+\tau)}|\mathcal{F}_t}(z) \approx \left(\phi^{\mathbb{Q}}_{\log \frac{S_T}{F(t,T)}|\mathcal{F}_t}(z)\right)^{\frac{\tau}{T-t}} \tag{3}$$

References

[1] Martin Baxter. Hedging in financial markets. *ASTIN Bulletin*, 28(1), 1998.

[2] Douglas Breeden and Robert Litzenberger. Prices of state-contingent claims implicit in option prices. *Journal of Business*, 51:621–652, 1978.

[3] Jim Gatheral. A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives. Global Derivatives & Risk Management presentation, Madrid, May 2004.

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[2]excess to funding.