Risk & Asset Allocation
Case Solution for Spring 1

John A. Dodson

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Say X is the invariant random variable that will drive market prices $P_{T+\tau} \triangleq g(X; p_T)$ between today, time T, and the next decision date, time $T+\tau$ with an investment horizon of τ (all time measured in years) and say that we have used historical data to estimate the parameters of the characterization of the random variable $\tilde{X}$ under a different (usually shorter) horizon, $\tilde{\tau}$; e.g., from a timeseries sample $(p_T, p_{T-\tilde{\tau}}, p_{T-2\tilde{\tau}}, \ldots)$.

Furthermore, say our invariant mapping is additive in the sense that

$$g\left(\tilde{x}_1; g\left(\tilde{x}_2, p_{T-2\tilde{\tau}}\right)\right) = g\left(\tilde{x}_1 + \tilde{x}_2; p_{T-2\tilde{\tau}}\right)$$

which is true, for example, with the continuous version of total return (although the simple version of total return is often an adequate approximation).

Then projecting X from $\tilde{X}$ essentially involves the transformation

$$X \triangleq \overbrace{\tilde{X} + \tilde{X} + \cdots}^{\frac{\tau}{\tilde{\tau}} \text{ i.i.d. copies}}$$

We know from the properties of characteristic functions that as long as increments are independent and identically distributed,

$$\phi_X(\omega) = \phi_{\tilde{X}}(\omega)^{\frac{\tau}{\tilde{\tau}}}$$

If the first two moments exist, we also know that

$$\begin{aligned}
\operatorname{E} X &= -i \left. \frac{d\phi_X}{d\omega'} \right|_0 \\
&= -i \frac{\tau}{\tilde{\tau}} \phi_{\tilde{X}}^{\frac{\tau}{\tilde{\tau}}-1} \left. \frac{d\phi_{\tilde{X}}}{d\omega'} \right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \operatorname{E} \tilde{X}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{E}\left(XX'\right) &= - \left. \frac{d^2\phi_X}{d\omega'\, d\omega} \right|_0 \\
&= -\frac{\tau}{\tilde{\tau}} \left(\frac{\tau}{\tilde{\tau}} - 1\right) \left(\phi_{\tilde{X}}\right)^{\frac{\tau}{\tilde{\tau}}-2} \frac{d\phi_{\tilde{X}}}{d\omega'} \frac{d\phi_{\tilde{X}}}{d\omega} - \frac{\tau}{\tilde{\tau}} \left(\phi_{\tilde{X}}\right)^{\frac{\tau}{\tilde{\tau}}-1} \left. \frac{d^2\phi_{\tilde{X}}}{d\omega'\, d\omega} \right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \left(\frac{\tau}{\tilde{\tau}} - 1\right) \operatorname{E} \tilde{X} \operatorname{E} \tilde{X}' + \frac{\tau}{\tilde{\tau}} \operatorname{E}\left(\tilde{X}\tilde{X}'\right) \\
&= \operatorname{E} X \operatorname{E} X' + \frac{\tau}{\tilde{\tau}} \left(\operatorname{E}\left(\tilde{X}\tilde{X}'\right) - \operatorname{E} \tilde{X} \operatorname{E} \tilde{X}'\right)
\end{aligned}$$

Since the covariance is defined as

$$\operatorname{cov}\tilde{X} = \operatorname{E}\left(XX'\right) - \operatorname{E}X\operatorname{E}X'$$

we have that

$$\operatorname{cov}X = \tfrac{\tau}{\tilde{\tau}}\operatorname{cov}\tilde{X}$$

Furthermore, since

$$\operatorname{std}\tilde{X} = \operatorname{diag}\sqrt{\operatorname{diag}\operatorname{diag}\operatorname{cov}\tilde{X}}$$

we have the "square-root rule" for time-scaling market invariants.

$$\operatorname{std}X = \sqrt{\tfrac{\tau}{\tilde{\tau}}}\operatorname{std}\tilde{X}$$

This is valid regardless of the distribution of $\tilde{X}$ (as long as it has two moments).

- Note that in general X will not belong to the same family of random variables as $\tilde{X}$.
- Further note that the presence of GARCH effects violate the i.i.d. assumption implicit in Meucci's definition of an invariant. Review the notes from last term about how variance scales in this setting.