Risk & Asset Allocation
Case for Spring 2

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This case previews of upcoming sessions in order to demonstrate the concept of an allocation-implied prior model. Let us examine the equilibrium allocation under exponential utility and normal markets.

Model

For a portfolio $\alpha_0 \# \alpha$ (α_0 represents cash) with net asset value

$$w = \alpha_0 + \alpha^\intercal p \tag{1}$$

the gain/loss over τ years would be

$$\Psi = \alpha_0 r\tau + \alpha^\intercal M \tag{2}$$

where the market vector is

$$M = P - p \tag{3}$$

with P the random variable for asset prices, including any cashflows, $\tau > 0$ years in the future and r the (simple-interest) return on cash.

Let us assume that the market vector is normal,

$$M \sim \mathcal{N}\left(\mu\,\tau, \Sigma\,\tau\right) \tag{4}$$

and that the preferences of the representative agent are described by exponential utility

$$u(\psi) = -e^{-\frac{\psi}{\zeta}} \tag{5}$$

with absolute risk aversion $1/\zeta > 0$.

Let us consider the portfolios that satisfy a wealth constraint $w^\star$ and maximize expected utility.

$$\begin{aligned} \operatorname{E} u(\Psi) &= -e^{-\frac{\alpha_0}{\zeta} r\tau} \operatorname{E} e^{-\frac{\alpha^\intercal}{\zeta} M} & (6) \\ &= -e^{-\frac{w^\star - \alpha^\intercal p}{\zeta} r\tau - \frac{\alpha^\intercal}{\zeta}\mu\tau + \frac{1}{2}\frac{\alpha^\intercal}{\zeta}\Sigma\tau\frac{\alpha}{\zeta}} & (7) \end{aligned}$$

So an optimal portfolio satisfies

$$\alpha^\star \in \arg\max_\alpha \alpha^\intercal(\mu - rp) - \frac{1}{2\zeta}\alpha^\intercal \Sigma \alpha \tag{8}$$

If the covariance is positive-definite, $\Sigma > 0$ (which it would not be if cash were included in the market vector), the first-order condition on the optimal portfolio is

$$\boxed{\mu = rp + \frac{1}{\zeta}\Sigma\alpha^\star} \tag{9}$$

This relationship, linking the market characterization to the investor utility and the optimal portfolio, can be the basis for an allocation-implied prior.

Notice that

$$\mathrm{E}\,\Psi^\star = w^\star r\tau + \frac{1}{\zeta}\operatorname{var}\Psi^\star \tag{10}$$

and more generally that

$$\mathrm{E}\,\Psi = wr\tau + \frac{1}{\zeta}\operatorname{cov}\left(\Psi, \Psi^\star\right) \tag{11}$$

$$= wr\tau + \frac{\operatorname{cov}\left(\Psi, \Psi^\star\right)}{\operatorname{var}\Psi^\star}\left(\mathrm{E}\,\Psi^\star - w^\star r\tau\right) \tag{12}$$

This is more recognizable as

$$\mathrm{E}\,\frac{\Psi}{w\tau} = r + \frac{\operatorname{cov}\left(\frac{\Psi}{w\tau}, \frac{\Psi^\star}{w^\star\tau}\right)}{\operatorname{var}\frac{\Psi^\star}{w^\star\tau}}\left(\mathrm{E}\,\frac{\Psi^\star}{w^\star\tau} - r\right) \tag{13}$$

where the coefficient is akin to "beta" in the capital asset pricing model.

Consider a portfolio consisting of a single share of the i-th stock.

$$\frac{\Psi}{w} = \frac{P_i}{p_i} - 1 \tag{14}$$

Hence

$$\mathrm{E}\,P_i = p_i(1 + r\tau) + \lambda\operatorname{cor}\left(P_i, \Psi^\star\right)\sqrt{\tau\operatorname{var}P_i} \tag{15}$$

where

$$\lambda \triangleq \frac{\sqrt{\alpha^{\star\intercal}\Sigma\alpha^\star}}{\zeta} \tag{16}$$

with dimensions $\mathrm{yr}^{-1/2}$ is termed the "market price of risk" and notably depends on neither the asset nor the investment horizon.

In particular, the expected value of the (simple) return on the i-th asset is

$$\bar{R}_i \triangleq r + \lambda\operatorname{cor}\left(P_i, \Psi^\star\right)\sqrt{\frac{\operatorname{var}P_i}{p_i^2\tau}} \tag{17}$$

whereby

$$\mathrm{E}\,P_i = p_i\left(1 + \bar{R}_i\tau\right) \tag{18}$$