Risk & Asset Allocation (Spring)
Case for Week 3

John A. Dodson

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Let us consider the expected shortfall index of satisfaction for a very simple portfolio: α shares in an asset whose value today is $p > 0$ and whose horizon value P is lognormal.

Let us assume that the objective measure is profit; therefore in Meucci's notation, we have

$$\begin{aligned}\Psi_\alpha &= \alpha M \\ &= \alpha\left(P - p\right) \\ &= \alpha\left(g(X) - p\right) \\ &= \alpha p\left(e^X - 1\right)\end{aligned}$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean μ and variance $\Sigma > 0$. The index of satisfaction is

$$\mathcal{S}(\alpha) = \frac{1}{1-c}\int_0^{1-c} Q_{\Psi_\alpha}(q)\,dq$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$\begin{aligned}F_{\Psi_\alpha}(z) &= \mathrm{P}\left\{\Psi_\alpha < z\right\} \\ &= \mathrm{P}\left\{\alpha p\left(e^X - 1\right) < z\right\} \\ &= \mathrm{P}\left\{X \operatorname{sgn}\alpha < \log\left(1 + \frac{z}{\alpha p}\right)\operatorname{sgn}\alpha\right\} \\ &= \mathrm{P}\left\{\frac{X-\mu}{\sqrt{\Sigma}}\operatorname{sgn}\alpha < \frac{\log\left(1+\frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma}}\operatorname{sgn}\alpha\right\} \\ &= \Phi\left(\frac{\log\left(1+\frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma}\operatorname{sgn}\alpha}\right)\end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the CDF, is therefore

$$Q_{\Psi_\alpha}(q) = \alpha p \left(e^{\mu + \operatorname{sgn}\alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$\begin{aligned}
\mathcal{S}(\alpha) &= \frac{1}{1-c} \int_0^{1-c} \alpha p \left(e^{\mu + \operatorname{sgn}\alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) \, dq \\
&= \alpha p \left(\frac{1}{1-c} \int_0^{1-c} e^{\mu + \operatorname{sgn}\alpha \sqrt{\Sigma} \Phi^{-1}(q)} \, dq - 1 \right) \\
&= \alpha p \left(\frac{1}{1-c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \operatorname{sgn}\alpha \sqrt{\Sigma} z} \phi(z) \, dz - 1 \right)
\end{aligned}$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \operatorname{sgn}\alpha \sqrt{\Sigma} z} \phi(z) = e^{\mu + \frac{1}{2}\Sigma} \phi\left(z - \operatorname{sgn}\alpha \sqrt{\Sigma} \right)$$

we have the final result,

$$\mathcal{S}(\alpha) = \alpha p \left(e^{\mu + \frac{1}{2}\Sigma} \frac{1}{1-c} \Phi\left(\Phi^{-1}(1-c) - \operatorname{sgn}\alpha \sqrt{\Sigma} \right) - 1 \right) \tag{1}$$

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$\mathcal{S}(\alpha) \approx \alpha p \left(\mu - \operatorname{sgn}\alpha \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \sqrt{\Sigma} \right) \tag{2}$$

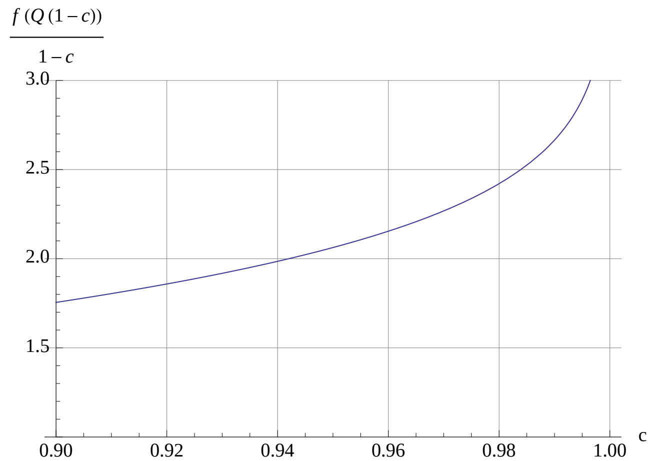
Let us spend a moment interpreting this. An investor will be more satisfied to be long ($\alpha > 0$) if the asset has a positive expected return ($\mu > 0$), and short ($\alpha < 0$) if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance diminishes satisfaction for any non-zero position.

This all seems quite reasonable for a rational index of satisfaction.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the objective Ψ_α. In a Delta-Gamma setting, we can replace the objective by the quadratic

$$\Psi_\alpha = \alpha p \left(e^X - 1 \right) \approx \alpha p \left(X + \tfrac{1}{2} X^2 \right)$$



hence $\Theta_\alpha = 0$, $\Delta_\alpha = \alpha p$, and $\Gamma_\alpha = \alpha p$. Let us define a new objective to represent this approximation.

$$\Xi_\alpha = \alpha p \left(X + \tfrac{1}{2} X^2 \right)$$

Is is straight-forward to work out that the first several central moments of this are

$$\begin{aligned}
\mathrm{E}\left(\Xi_\alpha\right) &= \alpha p \left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma \right) \\
\mathrm{Sd}\left(\Xi_\alpha\right) &= |\alpha| p \sqrt{\Sigma} \sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma} \\
\mathrm{Sk}\left(\Xi_\alpha\right) &= 3 \operatorname{sgn} \alpha \sqrt{\Sigma} \frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{\left((1+\mu)^2 + \tfrac{1}{2}\Sigma \right)^{3/2}}
\end{aligned}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$\mathcal{S}(\alpha) \approx \mathrm{E}\left(\Xi_\alpha\right) + \mathrm{Sd}\left(\Xi_\alpha\right) \left(z_1 + \frac{z_2 - 1}{6} \mathrm{Sk}\left(\Xi_\alpha\right) \right)$$

with coefficients

$$\begin{aligned}
z_1 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)\, dq = -\frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \\
z_2 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)^2\, dq = 1 - \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \Phi^{-1}(1-c)
\end{aligned}$$

depending on the confidence level $c < 1$[1].

Putting this together, we get a third expression for the index of satisfaction.

$$\begin{aligned}
\mathcal{S}(\alpha) \approx \alpha p \left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma \right) - |\alpha| p \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \sqrt{\Sigma} \\
\cdot \left(\sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma} + \tfrac{1}{2} \operatorname{sgn} \alpha \frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{(1+\mu)^2 + \tfrac{1}{2}\Sigma} \Phi^{-1}(1-c)\sqrt{\Sigma} \right) \qquad (3)
\end{aligned}$$

This result agrees with (2) to lowest order in μ and Σ.

[1] The trick to these integrals is to realize that $\phi'(z) = -z\phi(z)$.

4 Modeling Default

Our horizon asset value P is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$\Psi'_\alpha = \alpha p \left(Y e^X - 1 \right)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before[2], but now we add an independent default indicator $Y \sim \mathrm{Bern}(1-h)$ for default probability h.

[2] Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.