

Risk & Asset Allocation

Case for Spring 2

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This case previews of upcoming sessions in order to demonstrate the concept of an allocation-implied prior model. Let us examine the equilibrium allocation under exponential utility and normal markets.

Model

For a portfolio $\alpha_0 + \alpha$ (α_0 represents cash) with net asset value

$$w = \alpha_0 + \alpha^\top p \quad (1)$$

the gain/loss over τ years would be

$$\Psi = \alpha_0 r \tau + \alpha^\top M \quad (2)$$

where the market vector is

$$M = P - p \quad (3)$$

with P the random variable for asset prices, including any cashflows, $\tau > 0$ years in the future and r the (simple-interest) return on cash.

Let us assume that the market vector is normal,

$$M \sim \mathcal{N}(\mu \tau, \Sigma \tau) \quad (4)$$

and that the preferences of the representative agent are described by exponential utility

$$u(\psi) = -e^{-\frac{\psi}{\zeta}} \quad (5)$$

with absolute risk aversion $1/\zeta > 0$.

Let us consider the portfolios that satisfy a wealth constraint w^* and maximize expected utility.

$$\mathbb{E} u(\Psi) = -e^{-\frac{\alpha_0}{\zeta} r \tau} \mathbb{E} e^{-\frac{\alpha^\top}{\zeta} M} \quad (6)$$

$$= -e^{-\frac{w^* - \alpha^\top p}{\zeta} r \tau - \frac{\alpha^\top}{\zeta} \mu \tau + \frac{1}{2} \frac{\alpha^\top}{\zeta} \Sigma \tau \frac{\alpha}{\zeta}} \quad (7)$$

So an optimal portfolio satisfies

$$\alpha^* \in \arg \max_{\alpha} \alpha^\top (\mu - r p) - \frac{1}{2\zeta} \alpha^\top \Sigma \alpha \quad (8)$$

If the covariance is positive-definite, $\Sigma > 0$ (which it would not be if cash were included in the market vector), the first-order condition on the optimal portfolio is

$$\boxed{\mu = rp + \frac{1}{\zeta} \Sigma \alpha^*} \quad (9)$$

This relationship, linking the market characterization to the investor utility and the optimal portfolio, can be the basis for an allocation-implied prior.

Notice that

$$E \Psi^* = w^* r \tau + \frac{1}{\zeta} \text{var} \Psi^* \quad (10)$$

and more generally that

$$E \Psi = w r \tau + \frac{1}{\zeta} \text{cov}(\Psi, \Psi^*) \quad (11)$$

$$= w r \tau + \frac{\text{cov}(\Psi, \Psi^*)}{\text{var} \Psi^*} (E \Psi^* - w^* r \tau) \quad (12)$$

This is more recognizable as

$$E \frac{\Psi}{w \tau} = r + \frac{\text{cov}\left(\frac{\Psi}{w \tau}, \frac{\Psi^*}{w^* \tau}\right)}{\text{var} \frac{\Psi^*}{w^* \tau}} \left(E \frac{\Psi^*}{w^* \tau} - r \right) \quad (13)$$

where the coefficient is akin to “beta” in the capital asset pricing model.

Consider a portfolio consisting of a single share of the i -th stock.

$$\frac{\Psi}{w} = \frac{P_i}{p_i} - 1 \quad (14)$$

Hence

$$E P_i = p_i (1 + r \tau) + \lambda \text{cor}(P_i, \Psi^*) \sqrt{\tau \text{var} P_i} \quad (15)$$

where

$$\lambda \triangleq \frac{\sqrt{\alpha^{*\top} \Sigma \alpha^*}}{\zeta} \quad (16)$$

with dimensions $\text{yr}^{-1/2}$ is termed the “market price of risk” and notably depends on neither the asset nor the investment horizon.

In particular, the expected value of the (simple) return on the i -th asset is

$$\bar{R}_i \triangleq r + \lambda \text{cor}(P_i, \Psi^*) \sqrt{\frac{\text{var} P_i}{p_i^2 \tau}} \quad (17)$$

whereby

$$E P_i = p_i (1 + \bar{R}_i \tau) \quad (18)$$