A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, September 17.

**Problems**

This problem is about tail risk and hedging. Consider a portfolio with profit/loss
\[ \Pi = e^{X_1} - e^{X_2} \]
where \( X = (X_1, X_2)' \) is a random variable in \( \mathbb{R}^2 \) representing the assets’ log-returns

Let’s fix the marginal distributions,
\[
X_1 \sim \mathcal{N}(+0.001, 0.02^2) \\
X_2 \sim \mathcal{N}(-0.001, 0.02^2)
\]

and concordance in terms of Kendall’s tau
\[ \tau = 0.7 \]

Let’s consider two different copulas: the Gaussian copula that we discussed at lecture, and the Student’s-\( t_4 \) copula.

The \( t_4 \) copula with parameter \(-1 < \theta < 1\) for \( u \in (0, 1)^2 \) has the density
\[
f(u) = \frac{2\pi}{\sqrt{1 - \theta^2}} \left( \frac{1}{4(1 - \theta^2) + Q(u_1)^2 - 2\theta Q(u_1) Q(u_2) + Q(u_2)^2} \right)^{5/2}
\]
in terms of
\[
Q(p) = 2 \text{sgn} \left( p - \frac{1}{2} \right) \sqrt{\frac{1}{4p(1-p)}} \cos \left( \frac{1}{3} \tan^{-1} \sqrt{\frac{1}{4p(1-p)} - 1} \right) - 1
\]
which happens to be the quantile function of the Student’s-\( t_4 \).

1. What is \( E \Pi \)? (2 points)
2. What is \( \sqrt{\text{var} \Pi} \) under the Gaussian copula? (3 points)
3. What is \( \sqrt{\text{var} \Pi} \) under the Student’s-\( t_4 \) copula? (5 points)
Solution

I did the problem using MATLAB. First, I defined the relevant functions in a script for the copula densities, distributions, and Kendall’s tau’s: $E[4F_U(U_1, U_2) − 1]$.

% Gaussian copula
    cN=@(u1,u2,r)exp(-r/(1-r^2)*(r*erfcinv(2*u1).^2+r*erfcinv(2*u2).^2-2*erfcinv(2*u1).*erfcinv(2*u2)))/sqrt(1-r^2);
    CN=@(u1,u2,r)arrayfun(@(u1)dblquad(@(uu1,uu2)cN(uu1,uu2,r),eps,u1,eps,u2),u1);
    tauN=@(r)dblquad(@(u1,u2)(4*CN(u1,u2,r)-1).*cN(u1,u2,r),eps,1-eps,eps,1-eps);

% Student’s-t(4) copula
    Qt4=@(p)2*sign(p-1/2).*sqrt(cos(atan(sqrt(1./(4*p.*(1-p))-1))))/3;  
    ct4=@(u1,u2,r)2/9/pi*((1-r^2).*(4+Qt4(u1).^2).*(4+Qt4(u2).^2)).^3;  
    Ct4=@(u1,u2,r)arrayfun(@(u1)dblquad(@(uu1,uu2)ct4(uu1,uu2,r),eps,u1,eps,u2),u1);
    taut4=@(r)dblquad(@(u1,u2)(4*Ct4(u1,u2,r)-1).*ct4(u1,u2,r),eps,1-eps,eps,1-eps);

A few notes about MATLAB: dblquad() needs the integrand’s first argument to accept arrays. You can use arrayfun() to facilitate that where necessary. Also, because the copula density may be undefined on the boundary of $(0,1)^2$, use eps (the smallest positive floating point value) or a similar value to define the integration limits just inside of the boundaries.

We know that the Kendall’s tau for the Gaussian copula is

$$\tau = \frac{2}{\pi} \sin^{-1} \rho$$

and you can verify that with $\rho = \sin \left(\frac{\pi}{2} \times 0.7\right)$, you get a Kendall’s tau of 0.7. It turns out that this “pseudo-correlation” result is true for all elliptical copulas, including the Student’s-t; and in fact can can verify that

```matlab
>> r=sin(pi/2*.7)
  r =
      0.8910
  >> taut4(r)
  ans =
      0.7000
  >> tauN(r)
  ans =
      0.7000
```
Next, we define the marginal characterizations. Since they are the same, we only need to define one

\[
\begin{align*}
\text{fN} &= \exp(-z*z/2)/\sqrt{2\pi}; \\
\text{FN} &= (1+\text{erf}(z/\sqrt{2}))/2; \\
\text{QN} &= \sqrt{2} \times \text{erfinv}(2*p-1); \\
\end{align*}
\]

\[
\begin{align*}
f1 &= \text{fN}((x-0.001)/0.02)/0.02; \\
F1 &= \text{FN}((x-0.001)/0.02); \\
Q1 &= \text{QN}(p)0.001+0.02*\text{QN}(p); \\
f2 &= \text{fN}((x+0.001)/0.02)/0.02; \\
F2 &= \text{FN}((x+0.001)/0.02); \\
Q2 &= \text{QN}(p)0.001+0.02*\text{QN}(p); \\
\end{align*}
\]

We can now combine the marginals with the copula to get the joint densities for the two problems.

\[
\begin{align*}
density_N &= \text{cN}(F1(x1),F2(x2),r) \times f1(x1) \times f2(x2); \\
density_t4 &= \text{ct4}(F1(x1),F2(x2),r) \times f1(x1) \times f2(x2); \\
\end{align*}
\]

Finally, we can evaluate the moments of the profit and solve the problem.

\[
\begin{align*}
\text{profit} &= \exp(x1-x2); \\
\text{profit} &= \text{exp}(x1)-\text{exp}(x2); \\
\end{align*}
\]

\[
\begin{align*}
\text{M1}_N &= \text{dblquad}(\text{profit}(x1,x2) \times \text{density}_N(x1,x2)\ldots, Q1(1E-6), Q1(1E-6), Q2(1E-6), Q2(1E-6)); \\
\text{M1}_t4 &= \text{dblquad}(\text{profit}(x1,x2) \times \text{density}_{t4}(x1,x2)\ldots, Q1(1E-6), Q1(1E-6), Q2(1E-6), Q2(1E-6)); \\
\text{std}_N &= \sqrt{M2N-M1N^2}; \\
\text{M1}_t &= \text{dblquad}(\text{profit}(x1,x2) \times \text{density}_{t4}(x1,x2)\ldots, Q1(1E-6), Q1(1E-6), Q2(1E-6), Q2(1E-6)); \\
\text{M1}_t4 &= \text{dblquad}(\text{profit}(x1,x2) \times \text{density}_{t4}(x1,x2)\ldots, Q1(1E-6), Q1(1E-6), Q2(1E-6), Q2(1E-6)); \\
\text{std}_t4 &= \sqrt{M2t4-M1t4^2}; \\
\end{align*}
\]

1. In both cases, the expected value of the P/L is about 0.00200. The $t_4$ version is a tiny bit higher, but since the function is (almost) linear, we would not expect the expectation to depend much on the copula.

2. For the Gaussian copula, I get a P/L standard deviation of around 0.00931.

3. For the Student’s-$t_4$, I get a standard deviation of around 0.00952.

Notice that the standard deviation result is about 2% higher for the copula with tail dependence. This is not a huge difference, but it illustrates the point that assumptions about tail dependence can be important to your risk metrics.