Risk & Asset Allocation
Homework for Week 2

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A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, September 17.

Problems

This problem is about tail risk and hedging. Consider a portfolio with profit/loss

$$\Pi = e^{X_1} - e^{X_2}$$

where $X = (X_1, X_2)'$ is a random variable in $\mathbb{R}^2$ representing the assets' log-returns

Let's fix the marginal distributions,

$$X_1 \sim \mathcal{N}\left(+0.001, 0.02^2\right)$$
$$X_2 \sim \mathcal{N}\left(-0.001, 0.02^2\right)$$

and concordance in terms of Kendall's tau

$$\tau = 0.7$$

Let's consider two different copulas: the Gaussian copula that we discussed at lecture, and the Student's-t_4 copula.

The t_4 copula with parameter $-1 < \theta < 1$ for $u \in (0,1)^2$ has the density

$$f(u) = \frac{\frac{2}{9\pi}\left(1-\theta^2\right)^{5/2}\left(4+Q\left(u_1\right)^2\right)^{5/2}\left(4+Q\left(u_2\right)^2\right)^{5/2}}{\left(4\left(1-\theta^2\right)+Q\left(u_1\right)^2-2\theta Q\left(u_1\right)Q\left(u_2\right)+Q\left(u_2\right)^2\right)^3}$$

in terms of

$$Q(p) = 2\,\mathrm{sgn}\left(p-\tfrac{1}{2}\right)\sqrt{\sqrt{\frac{1}{4p(1-p)}}\cos\left(\tfrac{1}{3}\tan^{-1}\sqrt{\frac{1}{4p(1-p)}-1}\right)-1}$$

which happens to be the quantile function of the Student's-t_4.

1. What is $\mathrm{E}\,\Pi$? **(2 points)**
2. What is $\sqrt{\mathrm{var}\,\Pi}$ under the Gaussian copula? **(3 points)**
3. What is $\sqrt{\mathrm{var}\,\Pi}$ under the Student's-t_4 copula? **(5 points)**

Solution

I did the problem using MATLAB.

First, I defined the relevant functions in a script for the copula densities, distributions, and Kendall's tau's: $\mathrm{E}\left[4F_U(U_1,U_2)-1\right]$.

```
% Gaussian copula
cN=@(u1,u2,r)exp(-r/(1-r^2)*(r*erfcinv(2*u1).^2+r*erfcinv(2*u2).^2 ...
    -2*erfcinv(2*u1).*erfcinv(2*u2)))/sqrt(1-r^2);
CN=@(u1,u2,r)arrayfun(@(u1)...
    dblquad(@(uu1,uu2)cN(uu1,uu2,r),eps,u1,eps,u2),u1);
tauN=@(r)dblquad(@(u1,u2)...
     (4*CN(u1,u2,r)-1).*cN(u1,u2,r),eps,1-eps,eps,1-eps);

% Student's-t(4) copula
Qt4=@(p)2*sign(p-1/2).*sqrt(cos(atan(sqrt(1 ./(4*p.*(1-p))-1))/3)...
    ./sqrt(4*p.*(1-p))-1);
ct4=@(u1,u2,r)2/9/pi*((1-r^2).*(4+Qt4(u1).^2).*(4+Qt4(u2).^2)).^(5/2)...
    ./(4*(1-r^2)+Qt4(u1).^2-2*r*Qt4(u1).*Qt4(u2)+Qt4(u2).^2).^3;
Ct4=@(u1,u2,r)arrayfun(@(u1)...
    dblquad(@(uu1,uu2)ct4(uu1,uu2,r),eps,u1,eps,u2),u1);
taut4=@(r)dblquad(@(u1,u2)...
     (4*Ct4(u1,u2,r)-1).*ct4(u1,u2,r),eps,1-eps,eps,1-eps);
```

A few notes about MATLAB: `dblquad()` needs the integrand's first argument to accept arrays. You can use `arrayfun()` to facilitate that where necessary. Also, because the copula density may be undefined on the boundary of $(0,1)^2$, use `eps` (the smallest positive floating point value) or a similar value to define the integration limits just inside of the boundaries.

We know that the Kendall's tau for the Gaussian copula is

$$\tau = \frac{2}{\pi}\sin^{-1}\rho$$

and you can verify that with $\rho = \sin\left(\frac{\pi}{2}\times 0.7\right)$, you get a Kendall's tau of 0.7. It turns out that this "pseudo-correlation" result is true for all elliptical copulas, including the Student's-t; and in fact can can verify that

```
>> r=sin(pi/2*.7)

r =

    0.8910

>> taut4(r)

ans =

   0.7000

>> tauN(r)

ans =

   0.7000
```

Next, we define the marginal characterizations. Since they are the same, we only need to define one

```
% standard normal characterizations
fN=@(z)exp(-z.*z/2)/sqrt(2*pi);
FN=@(z)(1+erf(z/sqrt(2)))/2;
QN=@(p)sqrt(2)*erfinv(2*p-1);

% marginals
f1=@(x)fN((x-0.001)/0.02)/0.02;
F1=@(x)FN((x-0.001)/0.02);
Q1=@(p)0.001+0.02*QN(p);
f2=@(x)fN((x+0.001)/0.02)/0.02;
F2=@(x)FN((x+0.001)/0.02);
Q2=@(p)-0.001+0.02*QN(p);
```

We can now combine the marginals with the copula to get the joint densities for the two problems.

```
density_N=@(x1,x2)cN(F1(x1),F2(x2),r).*f1(x1).*f2(x2);
density_t4=@(x1,x2)ct4(F1(x1),F2(x2),r).*f1(x1).*f2(x2);
```

Finally, we can evaluate the moments of the profit and solve the problem.

```
% profit/loss
profit=@(x1,x2)exp(x1)-exp(x2);

% first result, Gaussian copula
M1_N=dblquad(@(x1,x2)profit(x1,x2).*density_N(x1,x2)...
    ,Q1(1E-6),Q1(1-1E-6),Q2(1E-6),Q2(1-1E-6));
M2_N=dblquad(@(x1,x2)profit(x1,x2).^2 .*density_N(x1,x2)...
    ,Q1(1E-6),Q1(1-1E-6),Q2(1E-6),Q2(1-1E-6));
std_N=sqrt(M2_N-M1_N^2);

% second result, Student's-t(4) copula
M1_t4=dblquad(@(x1,x2)profit(x1,x2).*density_t4(x1,x2)...
    ,Q1(1E-6),Q1(1-1E-6),Q1(1E-6),Q1(1-1E-6));
M2_t4=dblquad(@(x1,x2)profit(x1,x2).^2 .*density_t4(x1,x2)...
    ,Q1(1E-6),Q1(1-1E-6),Q1(1E-6),Q1(1-1E-6));
std_t4=sqrt(M2_t4-M1_t4^2);
```

1. In both cases, the expected value of the P/L is about 0.00200. The t_4 version is a tiny bit higher, but since the function is (almost) linear, we would not expect the expectation to depend much on the copula.

2. For the Gaussian copula, I get a P/L standard deviation of around 0.00931.

3. For the Student's-t_4, I get a standard deviation of around 0.00952.

Notice that the standard deviation result is about 2% higher for the copula with tail dependence. This is not a huge difference, but it illustrates the point that assumptions about tail dependence can be important to your risk metrics.