Risk & Asset Allocation
Homework for Week 3

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A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, September 24.

Problem

Let's explore the simple defaultable asset model I described in the first week. Say an asset price X at some future date is a Bernoulli mixture of a log-normal with fixed parameters $\sigma > 0$ and $\mu > 0$, with probability $e^{-\lambda}$, and a Dirac at zero, with probability $1 - e^{-\lambda}$ for fixed $\lambda \geq 0$.

1. What value of μ is required such that $\operatorname{E} X = Se^r$ for fixed S and r? **(5 points)**

2. For that value of μ, what is the value of $e^{-r} \operatorname{E} \max(0, S - X)$? **(5 points)**

   Hint: Use the tower property.

Solution

Let's write down the hierarchical model for the underlying price[1].

$$
\begin{aligned}
X|Y &\sim \begin{cases} \mathsf{LogN}(\mu, \sigma) & Y = \text{“no default”} \\ \delta(0) & Y = \text{“default”} \end{cases} \\
Y &\sim \mathsf{Bern}\left(1 - e^{-\lambda}\right)
\end{aligned}
$$

We can use the tower property to evaluate these expectations[2].

$$
\begin{aligned}
\operatorname{E} X &= \operatorname{E}\left[\operatorname{E} X|Y\right] \\
&= e^{-\lambda} \operatorname{E}\left[X|Y = \text{“no default”}\right] + \left(1 - e^{-\lambda}\right) \operatorname{E}\left[X|Y = \text{“default”}\right] \\
&= e^{-\lambda} \mu e^{\frac{1}{2}\sigma^2}
\end{aligned}
$$

[1] In the initial version, I had the Bernoulli probabilities reversed. This version is a little clearer.

[2] The slides and the text differ in the definition of μ. If you are using the definition in the text, you will have e^{μ} in place of μ.

hence

$$\mu = Se^{r+\lambda-\frac{1}{2}\sigma^2} \tag{1}$$

satisfies the no-arbitrage condition $\mathrm{E}\,X = Se^r$.

Now, let's value the put option contract.

$$\mathrm{E}\max(0, S-X) = e^{-\lambda} \int_0^S (S-x)\frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\log(x/\mu)^2/\sigma^2}\ dx + \left(1-e^{-\lambda}\right) S$$

The integral can be evaluated more easily by substituting

$$u = \frac{\log(x/\mu)}{\sigma} \qquad du = \frac{1}{\sigma x}\ dx$$

whereby

$$\mathrm{E}\max(0, S-X) = e^{-\lambda} \int_{-\infty}^{\log(S/\mu)/\sigma} (S-\mu\, e^{\sigma u})\, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\ du + \left(1-e^{-\lambda}\right) S$$

Using the notation $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2 du}$ for the standard normal distribution function, and the fact that

$$\sigma u - \tfrac{1}{2}u^2 = -\tfrac{1}{2}(u-\sigma)^2 + \tfrac{1}{2}\sigma^2$$

we get to

$$\begin{aligned}\mathrm{E}\max(0, S-X) = Se^{-\lambda}\Phi\left(\frac{\log(S/\mu)}{\sigma}\right) + S\left(1-e^{-\lambda}\right)& \\ -\,\mu\, e^{-\lambda+\frac{1}{2}\sigma^2} \int_{-\infty}^{\log(S/\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\sigma)^2}\ du&\end{aligned}$$

The remaining integral is just the distribution of a normal with mean σ and variance one, so

$$\mathrm{E}\max(0, S-X) = Se^{-\lambda}\Phi\left(\frac{\log(S/\mu)}{\sigma}\right) + S\left(1-e^{-\lambda}\right) - \mu\, e^{-\lambda+\frac{1}{2}\sigma^2}\Phi\left(\frac{\log(S/\mu)}{\sigma} - \sigma\right)$$

Substituting in the solution for μ and discounting, we get the result

$$e^{-r}\,\mathrm{E}\max(0, S-X) = Se^{-r-\lambda}\Phi\left(\tfrac{1}{2}\sigma - \frac{r+\lambda}{\sigma}\right) + S\left(e^{-r} - e^{-r-\lambda}\right) - S\Phi\left(-\tfrac{1}{2}\sigma - \frac{r+\lambda}{\sigma}\right)$$

Employing the symmetries of $\Phi(\cdot)$, we can simplify this somewhat to

$$e^{-r}\,\mathrm{E}\max(0, S-X) = S\Phi\left(\frac{r+\lambda}{\sigma} + \tfrac{1}{2}\sigma\right) - Se^{-r-\lambda}\Phi\left(\frac{r+\lambda}{\sigma} - \tfrac{1}{2}\sigma\right) + Se^{-r} - S$$

Discussion

The more general result for a put with strike price $K > 0$ is

$$e^{-r} \operatorname{E} \max(0, K - X) = \\ S\Phi\left(\frac{\log(S/K) + r + \lambda}{\sigma} + \tfrac{1}{2}\sigma\right) - Ke^{-r-\lambda}\Phi\left(\frac{\log(S/K) + r + \lambda}{\sigma} - \tfrac{1}{2}\sigma\right) \\ + Ke^{-r} - S \quad (2)$$

If we let $\lambda = 0$, we get the Black-Scholes result. It is conventional to use the Black-Scholes formula to quote option "implied volatility" even if the Black-Scholes assumptions do not hold. In this case, where we drop the assumption about the non-defaultability of the underlying, the implied volatility $\sigma_{BS}(K)$ is defined implicitly by

$$S\Phi\left(\frac{\log(S/K) + r + \lambda}{\sigma} + \tfrac{1}{2}\sigma\right) - Ke^{-r-\lambda}\Phi\left(\frac{\log(S/K) + r + \lambda}{\sigma} - \tfrac{1}{2}\sigma\right) \\ = S\Phi\left(\frac{\log(S/K) + r}{\sigma_{BS}(K)} + \tfrac{1}{2}\sigma_{BS}(K)\right) - Ke^{-r}\Phi\left(\frac{\log(S/K) + r}{\sigma_{BS}(K)} - \tfrac{1}{2}\sigma_{BS}(K)\right) \quad (3)$$

Note what this relationship says (at least for calls):

$$r + \lambda \mapsto r \qquad \Longrightarrow \qquad \sigma \mapsto \sigma_{BS}(K)$$

For example, with $r = 0$, $\sigma = 0.2$ and $\lambda = \{0, 0.005\}$ we get the implied volatility curve in Figure 1.

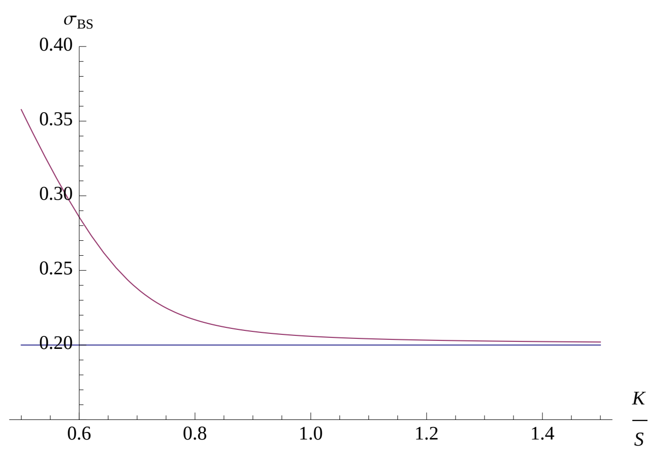

Figure 1: Implied volatility for $r = 0$, $\sigma = 0.2$, and $\lambda = \{0, 0.005\}$.

This pattern of higher implied volatilities for lower strike prices is typically observed in equity securities options and is termed "skewness" (no relation to the statistical term). While this is probably not a complete theory, we see that some skewness emerges naturally from even a simple, static description of the defaultability of the underlying.