Risk & Asset Allocation (Spring)
Homework for Week 1

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Introduction

Treasury nominal[1] bonds pay coupons at a fixed rate in equal installments twice per year. The usual approach to valuing bonds is based on the net present value of these cashflows. The basis for this is a discounting function, which could be arbitrarily complicated. Faced with estimation, a dimension reduction is called for. There are several popular functional forms for the discounting function. We will consider the following:

$$d(T-t;\beta) = \exp\left\{-\beta_2(T-t) + \beta_3\frac{1-e^{-\beta_1(T-t)}}{\beta_1} - \left(\beta_4\frac{1-e^{-\beta_1(T-t)}}{\beta_1}\right)^2\right\} \tag{1}$$

If a new par bond with maturity date T has yield (equal to coupon) $y(t,T)$ on date t, then

$$1 = d(T-t;\beta(t)) + \tfrac{1}{2}y(t,T)\sum_{i=1}^{2(T-t)} d(i/2;\beta(t)) \tag{2}$$

can be used to define—or at least to fit—parameters $\beta(t)$ to this model. One approach to this is (non-linear) least-squares

$$\hat\beta(t) \triangleq \arg\min_\beta \sum_j \left(\frac{1-d\left(T_j-t;\beta\right)}{\frac{1}{2}\sum_{i=1}^{2(T_j-t)} d\left(i/2;\beta\right)} - y\left(t,T_j\right)\right)^2$$

where j indexes the yields to be fit.

Problem

A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, January 28.

1. Using the Federal Reserve's H.15 statistical release, available here:

   `http://www.federalreserve.gov/releases/h15/current/default.htm`

   fit parameters $\hat\beta(t)$ to $T_j - t = $ 1-, 2-, 3-, 5-, 7-, 10-, 20-, and 30-year constant maturity Treasury nominal yields $y\left(t,T_j\right)$ for $t =$ January 14. Keep in mind that these yields are quoted in percent. **(6 points)**

[1] "nominal" in contrast to "discount", "inflation-protected", or "registered interest and principal"

2. Make a table of the yields you used and the fitted yields you got with the parameters from above (based on (2)). If you are successful, these should be within about 5 basis points (5.E-4/yr) for each maturity. **(4 points)**

Solution

Below is the MATLAB I wrote to identify the parameters for the January 14 nominal Treasury yields.

```
>> tenor=[1 2 3 5 7 10 20 30]; % constant maturity Treasuries tenors
>> yield=[0.18 0.51 0.83 1.33 1.62 1.86 2.20 2.47]/100; % from Fed H.15 for 2015/01/14
>> disc=@(T,b)exp(-b(2)*T+b(3)*(1-exp(-b(1)*T))/b(1)-(b(4)*(1-exp(-b(1)*T))/b(1)).^2);
>> yld=@(T,b)arrayfun(@(T)2*(1-disc(T,b))/sum(disc(.5:.5:T,b)),T);
>> obj=@(b)sum(arrayfun(@(T,cpn)(yld(T,b)-cpn)^2,tenor,yield));
>> b0=fminsearch(obj,[.05 .05 .05 .05]);
```

The results I got for the parameters are here

Table 1: Least-squares (yield) parameters for January 14.

β_1	0.2136 / yr
β_2	0.0283 / yr
β_3	0.0318 / yr
β_4	0.0473 / yr

and the fitted yields are here

Table 2: Fit and residuals for January 14.

tenor	data	fit	error
1 yr	0.0018 / yr	0.0016 / yr	-0.0002 / yr
2 yr	0.0051 / yr	0.0055 / yr	+0.0003 / yr
3 yr	0.0083 / yr	0.0085 / yr	+0.0002 / yr
5 yr	0.0133 / yr	0.0130 / yr	-0.0003 / yr
7 yr	0.0162 / yr	0.0159 / yr	-0.0003 / yr
10 yr	0.0186 / yr	0.0187 / yr	+0.0001 / yr
20 yr	0.0220 / yr	0.0227 / yr	+0.0007 / yr
30 yr	0.0247 / yr	0.0242 / yr	-0.0005 / yr

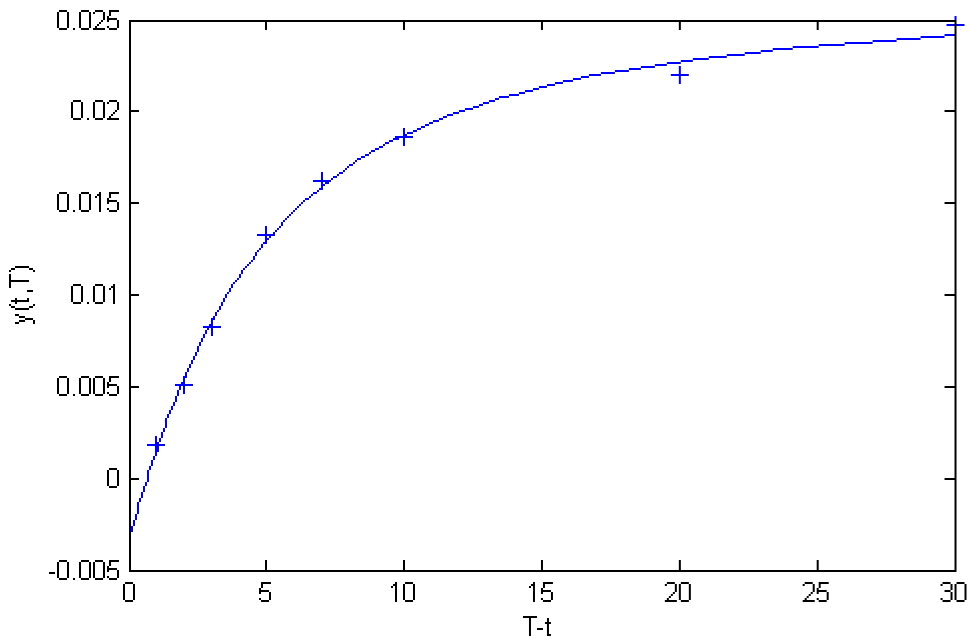
In order to plot a yield curve, I need to invert (2). I also need to extend its range to non-integer tenors. The usual way to do this is to introduce the concept of "clean price", whereby "accrued interest" on the current coupon is excluded. The net present value of a par bond is then par *plus accrued*.

$$1 + y(t,T)\left(\tfrac{1}{2}\lceil 2(T-t)\rceil - T + t\right) = d(T-t;\beta(t)) + \tfrac{1}{2}y(t,T)\sum_{i=0}^{\lceil 2(T-t)\rceil-1} d\left(T - i/2 - t;\beta(t)\right) \qquad (2')$$

$y(t,T)$ defined implicitly here is continuous in T (as long as $d(T-t)$ is for $T > t$ and $\lim_{t\nearrow T} d(T-t) = 1$) because both sides gap by $\frac{1}{2}y(t,T)$ when $T-t$ crosses a half-integer, in accordance with a coupon payment.

```
>> par_curve=@(T)arrayfun(@(T)(1-disc(T,b0))...
    /(T-ceil(2*T)/2+sum(disc(T-(0:ceil(2*T)-1)/2,b0))/2),T); % includes accrued
>> fplot(par_curve,[0 30]); hold on; plot(tenor,yield,'+'); hold off;
```

Figure 1: Nominal Treasuries par curve for January 14, 2015



Discussion

While described as strictly empirical, this model has a basis in theory. If the coefficients are observed to have a certain dynamic, namely $\beta_1, \beta_2, \beta_4$ constant and $\beta_3(t)$ an Ornstein-Uhlenbeck process, this model is equivalent to the Vasicek (1977) single-factor gaussian short-rate model:

$$d(T-t) = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \middle| \mathcal{F}_t \right]$$

where

$$\begin{aligned} dr_t &= \left(2\beta_4^2 + \beta_1\beta_2 - \beta_1 r_t\right) dt + 2\beta_4\sqrt{\beta_1}\, dW \\ r_t &= \beta_2 - \beta_3(t) \end{aligned}$$

with W a standard brownian motion under $\mathbb{Q}$.