Risk & Asset Allocation (Spring)
Homework for Week 2

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A solution to this problem is due at the beginning of the next session, which is 5:30 PM on February 4.

Problem

Say you have $N = 500$ historical return observations for some asset. Assume they are i.i.d. (perhaps a GARCH model has been fit). You wish to model the invariant with a normal.

- The sample mean of the data is -0.0027 and the sample variance is 0.00022.
- Your prior on this return is that the mean is $+0.005$ and the variance is 0.0002.

1. What is the standard error of the mean estimate based on the sample? **(3 points)**
2. How much pseudo-data do you need to add (this is $\lambda_0 = \nu_0$ in the notation of the lecture) in order for the posterior standard deviation of the mean to be no more than 0.0005? **(5 points)**

   (Note: the marginal posterior for the mean is a Student's-t with variance

   $$\operatorname{var} \mu = \frac{\sigma_N^2}{\lambda_N} \frac{\nu_N}{\nu_N - 2}$$

   in the notation of the lecture.)
3. What is the point estimate of the posterior mean? **(2 points)**

Solution

From last semester, we know the the standard error of the mean of a sample from $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is

$$\sqrt{\operatorname{var} \hat{\mu}} = \sqrt{\frac{\sigma^2}{N}}$$

Using the sample variance in place of σ^2, we get that the standard error of the mean is $\sqrt{0.00022/500} \approx 0.00066$.

Now we turn to calibrating the conjugate prior hyper-parameters to improve this standard error. We are given $\mu_0 = 0.005$ and $\sigma_0^2 = 0.0002$ directly. Let's let $\lambda_0 = \nu_0 \triangleq N'$ be defined implicitly as the smallest integer solution to

$$\frac{\sigma_N^2}{\lambda_N} \frac{\nu_N}{\nu_N - 2} \leq 0.0005^2 \tag{1}$$

where $\lambda_N = \nu_N = N' + N$ and

$$\mu_N = \frac{N'\mu_0 + N\hat{\mu}}{N' + N}$$
$$\sigma_N^2 = \frac{N'\left(\sigma_0^2 + \mu_0^2\right) + N\left(\hat{\mu}^2 + \hat{\sigma}^2\right)}{N' + N} - \mu_N^2$$

with the summary statistics $\hat{\mu} = -0.0027$ and $\hat{\sigma}^2 = 0.00022 \times (N-1)/N$.

Substituting these definitions into (1) gives a cubic inequality in N', which evaluates to

$$N' > 403.9$$

whose smallest integer solution is $N' = 404$.

There is an exact solution for this, but I used the following MATLAB script:

```
>> N=500;
>> t1=-2.7E-3*N;
>> t2=N*(2.2E-4*(N-1)/N+(t1/N)^2); % note adaption for unbiased sample variance
>> mu0=5E-3;
>> sigmaSqr0=2E-4;
>> nuN=@(nu0)nu0+N;
>> muN=@(nu0)(nu0*mu0+t1)/nuN(nu0);
>> sigmaSqrN=@(nu0)(nu0*sigmaSqr0+nu0*mu0^2+t2-nuN(nu0)*muN(nu0)^2)/nuN(nu0);
>> fzero(@(nu0)sigmaSqrN(nu0)/(nuN(nu0)-2)-5E-4^2,N)

ans =

  403.8990

>> muN(404)

ans =

    7.4114e-04
```

where in the last step we have used the result that

$$\mathrm{E}\,\mu = \mu_N \approx +0.0007$$

The Student's-t distribution is symmetric, so the mean is also the mode, in case we wanted to use the maximum likelihood point estimator instead.