Risk & Asset Allocation (Spring)
Homework for Week 3

John A. Dodson

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A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, February 11.

Problem

Assume that the value for a portfolio over some analysis horizon is a random variable, Ψ, with a Generalized Pareto left tail partially defined by the distribution function

$$F_\Psi(\psi) \approx \begin{cases} \theta \left(1 + \xi \frac{\eta - \psi}{\beta}\right)^{-1/\xi} & \psi \leq \eta \\ ? & \text{otherwise} \end{cases}$$

with tail parameter $0 < \xi < 1$, scale parameter $\beta > 0$, threshold parameter η, and tail mass $0 < \theta < 1$.

- Provide expressions for the value-at-risk **(2 points)** and expected shortfall **(3 points)** at confidence level c with $1 - \theta \leq c < 1$ in terms of the parameters of this partial characterization.
- Let's say we add some cash to the portfolio, so that the new horizon value is $\Psi' \triangleq \Psi + \psi_b$ for some fixed $\psi_b \geq 0$. What is the new distribution function $F_{\Psi'}(\cdot)$ **(3 points)**? Provide expressions for the value-at-risk **(1 point)** and expected shortfall for the same confidence as above for this new portfolio. Are these metrics translation-invariant **(1 point)**?

Solution

Value-at-risk is based on the quantile function, which is the inverse of the distribution function. For a sufficiently high confidence level, we can invert $F_\Psi(\cdot)$ to get

$$\text{VaR}_c \triangleq Q_\Psi(1-c) = \eta - \frac{\beta}{\xi}\left(\left(\frac{\theta}{1-c}\right)^{\xi} - 1\right) \tag{1}$$

To evaluate the expected shortfall, we can use the definition

$$\text{ES}_c \triangleq \frac{1}{1-c}\int_0^{1-c} Q_\Psi(p)\, dp$$

to get

$$\mathrm{ES}_c = \eta - \frac{\beta}{\xi}\left(\frac{1}{1-\xi}\left(\frac{\theta}{1-c}\right)^{\xi} - 1\right) \tag{2}$$

As an aside, McNeil *et al.* points out that for high confidence levels, $\eta + \frac{\beta}{\xi}$ is negligible; so

$$\lim_{c\uparrow 1} \frac{\mathrm{ES}_c}{\mathrm{VaR}_c} = \frac{1}{1-\xi}$$

This is a nice way to think about the interpretation of the tail parameter, ξ. Expected shortfall is larger than value-at-risk; this relationship tells us by how much. Also, a normal tail has $\xi = 0$, so this result highlights how expected shortfall and value-at-risk diverge for non-normal tails.

With cash included, the new distribution function come from the old simply by substituting the argument $\psi \mapsto \psi - \psi_b$.

$$F_{\Psi'}(\psi) \approx \begin{cases} \theta\left(1 + \xi\frac{\eta - \psi + \psi_b}{\beta}\right)^{-1/\xi} & \psi \le \eta + \psi_b \\ ? & \text{otherwise} \end{cases}$$

Notice that this is equivalent to

$$F_{\Psi'}(\psi) \approx \begin{cases} \theta\left(1 + \xi\frac{\eta' - \psi}{\beta}\right)^{-1/\xi} & \psi \le \eta' \\ ? & \text{otherwise} \end{cases}$$

where $\eta' \triangleq \eta + \psi_b$. That is, the addition of cash is reflected simply by shifting the threshold parameter.

Since equations (1) and (2) for value-at-risk and expected shortfall are additive in η, these metrics as indexes of satisfaction are translation invariant.