Risk & Asset Allocation (Spring)
Homework for Week 4

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A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, February 18.

The goal of this problem is to work through the two-step approach to determine the optimal portfolio in an analytic setting.

Assume a single affine constraint (e.g. the wealth constraint), an investor objective linear in the market vector (e.g. wealth or profit), and a sufficiently short analysis horizon such that the skewness of the market vector is negligible.

For an objective defined by $\Psi_\alpha = \alpha' M$ and an affine constraint defined by $d'\alpha = c > 0$, the analytic solution to the optimal mean-variance portfolio satisfies

$$\alpha(\beta) = (1-\beta)\alpha_{MV} + \beta\alpha_{SR} \tag{1}$$

for $\beta \geq 0$, where

$$\alpha_{MV} = \frac{c\left(\mathsf{cov} M\right)^{-1} d}{d'\left(\mathsf{cov} M\right)^{-1} d}$$

$$\alpha_{SR} = \frac{c\left(\mathsf{cov} M\right)^{-1} \mathsf{E} M}{d'\left(\mathsf{cov} M\right)^{-1} \mathsf{E} M}$$

This is step one. Step two is to determine the level of β that maximizes the investor's index of satisfaction, which we will take to be the Cornish-Fisher expansion of the 95% expected shortfall.

$$\mathcal{S}(\alpha) = \mathsf{E}\,\Psi_\alpha + \sqrt{\mathsf{var}\,\Psi_\alpha}\left(\mathcal{I}\left[\phi\Phi^{-1}\right] + \tfrac{1}{6}\left(\mathcal{I}\left[\phi\left(\Phi^{-1}\right)^2\right] - 1\right)\mathsf{sk}\,\Psi_\alpha\right) \tag{2}$$

where Φ is the CDF of a standard normal random variable and ϕ is the coherent spectrum for $\mathrm{ES}_{0.95}$.

Note that

$$\mathcal{I}\left[\phi\Phi^{-1}\right] = \int_0^{0.05} \frac{\sqrt{2}\,\mathrm{erf}^{-1}(2p-1)}{0.05}\,dp \approx -2.0627\cdots$$

Problem

1. Solve
$$\beta^\star = \arg\max_{\beta\geq 0} \mathcal{S}\left(\alpha(\beta)\right)$$
Hint: $\Psi_{\alpha_{MV}}$ and $\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}$ are uncorrelated. **(6 points)**

2. Under what conditions is $\beta^\star$ finite? **(4 points)**

Solution

Let us assign $z_{0.95} = 2.0627\cdots$. The satisfaction is

$$\mathcal{S}(\alpha) = \alpha' \mathsf{E} M - z_{0.95}\sqrt{\alpha' \left(\mathsf{cov} M\right)\alpha}$$

Substituting in (1), we get that the optimal value for β is

$$\beta^{*} = \arg\max_{\beta \geq 0} \left(\alpha'_{MV} + \beta\left(\alpha'_{SR} - \alpha'_{MV}\right)\right) \mathsf{E} M \\ - z_{0.95}\sqrt{\left(\alpha'_{MV} + \beta\left(\alpha'_{SR} - \alpha'_{MV}\right)\right)\left(\mathsf{cov} M\right)\left(\alpha_{MV} + \beta\left(\alpha_{SR} - \alpha_{MV}\right)\right)}$$

Expressing this in terms of the moments of $\Psi_{\alpha_{SR}}$ and $\Psi_{\alpha_{MV}}$, and making use of the hint (and dropping the constant), this is the same as

$$\beta^{*} = \arg\max_{\beta \geq 0} \beta\, \mathrm{E}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right) - z_{0.95}\sqrt{\mathrm{var}\, \Psi_{\alpha_{MV}} + \beta^2\, \mathrm{var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)}$$

It is convenient at this stage to introduce the “information ratio”,

$$\gamma \triangleq \frac{\mathrm{E}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)}{\sqrt{\mathrm{var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)}} \tag{3}$$

Factoring out a (positive) constant, we get

$$\beta^{*} = \arg\max_{\beta \geq 0} \gamma\, \beta - z_{0.95}\sqrt{\frac{\mathrm{var}\, \Psi_{\alpha_{MV}}}{\mathrm{var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)} + \beta^2} \tag{4}$$

Notice at if $\gamma \geq z_{0.95}$, the objective of the maximization above becomes arbitrarily large as $\beta \to \infty$. Therefore there is no finite solution for the optimal portfolio if the satisfaction confidence level is too low. Also notice that if $\gamma \leq 0$, i.e. if the SR portfolio is not expected to out-perform the MV portfolio, the objective is strictly decreasing in β, so the optimal solution is $\alpha^{\star} = \alpha_{MV}$. For $0 < \gamma < z_{0.95}$, solving (4) is an elementary calculus problem.

In summary,

$$\beta^{\star} = \begin{cases} 0 & \gamma \leq 0 \\ \sqrt{\frac{\mathsf{var}\Psi_{\alpha_{MV}}}{\mathsf{var}\left(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}\right)} \frac{1}{\left(\frac{z_{0.95}}{\gamma}\right)^2 - 1}} & 0 < \gamma < z_{0.95} \\ +\infty & \gamma \geq z_{0.95} \end{cases} \tag{5}$$

Appendix: First-order condition

We are interested in the maximum of a real function of a single real variable of the form

$$f : x \mapsto f(x) = Ax - \sqrt{x^2 + B}$$

for constants $0 < A < 1$, $B \geq 0$ in the domain $x \geq 0$.

The first-order condition for a local maximum is

$$0 = f'(x^\star) = A - \frac{x^\star}{\sqrt{(x^\star)^2 + B}}$$

Since A and $x^\star$ are positive, if there is a solution, then

$$\begin{aligned} \sqrt{(x^\star)^2 + B} &= \frac{x^\star}{A} \\ (x^\star)^2 + B &= \frac{(x^\star)^2}{A^2} \\ B &= (x^\star)^2 \left(\frac{1}{A^2} - 1 \right) \end{aligned}$$

hence

$$\boxed{x^\star = \sqrt{\frac{B}{\frac{1}{A^2} - 1}}}$$

Note that $x^\star$ is increasing in both A and B and $x^\star = 0$ if either $A \to 0$ or $B = 0$.