

Quantitative Risk Management

Case for Week 3 (Spring)

John A. Dodson

February 3, 2016

Let us consider the expected shortfall index of satisfaction for a very simple portfolio: λ shares in an asset whose value today is $p > 0$ and whose horizon value P is lognormal.

Let us assume that the objective measure is mark-to-market profit; therefore in the text's notation, we have (apologies for the signs)

$$\begin{aligned} -L &= \lambda(P - p) \\ &= \lambda p(e^X - 1) \end{aligned}$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean μ and variance $\Sigma > 0$. The risk measure is

$$-\varrho(L) = -r_\varrho(\lambda) = \frac{1}{1-c} \int_0^{1-c} F_{-L}^{\leftarrow}(q) dq$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$\begin{aligned} F_{-L}(z) &= \mathbb{P}\{-L < z\} \\ &= \mathbb{P}\{\lambda p(e^X - 1) < z\} \\ &= \mathbb{P}\left\{X \operatorname{sgn} \lambda < \log\left(1 + \frac{z}{\lambda p}\right) \operatorname{sgn} \lambda\right\} \\ &= \mathbb{P}\left\{\frac{X - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \lambda < \frac{\log\left(1 + \frac{z}{\lambda p}\right) - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \lambda\right\} \\ &= \Phi\left(\frac{\log\left(1 + \frac{z}{\lambda p}\right) - \mu}{\sqrt{\Sigma} \operatorname{sgn} \lambda}\right) \end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the distribution function, is therefore

$$F_{-L}^{\leftarrow}(q) = \lambda p \left(e^{\mu + \text{sgn } \lambda \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$\begin{aligned} -r_{\varrho}(\lambda) &= \frac{1}{1-c} \int_0^{1-c} \lambda p \left(e^{\mu + \text{sgn } \lambda \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq \\ &= \lambda p \left(\frac{1}{1-c} \int_0^{1-c} e^{\mu + \text{sgn } \lambda \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right) \\ &= \lambda p \left(\frac{1}{1-c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn } \lambda \sqrt{\Sigma} z} \phi(z) dz - 1 \right) \end{aligned}$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \text{sgn } \lambda \sqrt{\Sigma} z} \phi(z) = e^{\mu + \frac{1}{2}\Sigma} \phi \left(z - \text{sgn } \lambda \sqrt{\Sigma} \right)$$

we have the final result,

$$-r_{\varrho}(\lambda) = \lambda p \left(e^{\mu + \frac{1}{2}\Sigma} \frac{1}{1-c} \Phi \left(\Phi^{-1}(1-c) - \text{sgn } \lambda \sqrt{\Sigma} \right) - 1 \right) \quad (1)$$

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$-r_{\varrho}(\lambda) \approx \lambda p \left(\mu - \text{sgn } \lambda \frac{\phi \left(\Phi^{-1}(1-c) \right)}{1-c} \sqrt{\Sigma} \right) \quad (2)$$

Let us spend a moment interpreting this. A long is less risky ($\lambda > 0$) if the asset has a positive expected return ($\mu > 0$), and a short ($\lambda < 0$) is less risky if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance increases risk for any non-zero position.

This all seems quite reasonable for a rational risk measure.

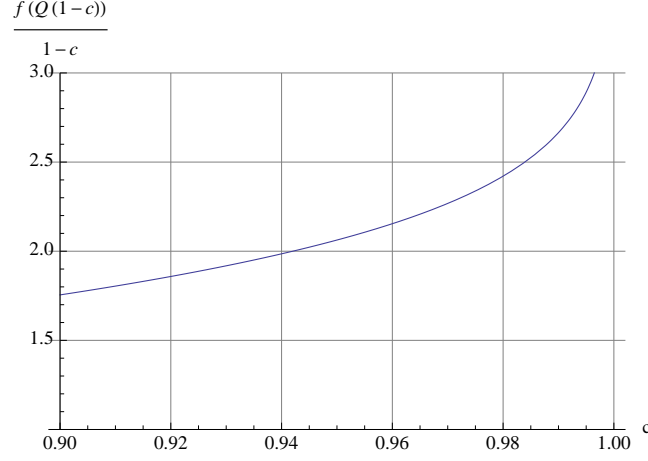
3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the loss. In a Delta-Gamma setting, we can replace the objective by the quadratic

$$-L = \lambda p \left(e^X - 1 \right) \approx \lambda p \left(X + \frac{1}{2} X^2 \right)$$

hence $\Theta_{\lambda} = 0$, $\Delta_{\lambda} = \lambda p$, and $\Gamma_{\lambda} = \lambda p$. Let us define a new objective to represent this approximation.

$$\Xi_{\lambda} = \lambda p \left(X + \frac{1}{2} X^2 \right)$$



Is is straight-forward to work out that the first several central moments of this are

$$\begin{aligned} E(\Xi_\lambda) &= \lambda p \left(\mu + \frac{1}{2}\mu^2 + \frac{1}{2}\Sigma \right) \\ Sd(\Xi_\lambda) &= |\lambda| p \sqrt{\Sigma} \sqrt{(1 + \mu)^2 + \frac{1}{2}\Sigma} \\ Sk(\Xi_\lambda) &= 3 \operatorname{sgn} \lambda \sqrt{\Sigma} \frac{(1 + \mu)^2 + \frac{1}{3}\Sigma}{\left((1 + \mu)^2 + \frac{1}{2}\Sigma \right)^{3/2}} \end{aligned}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$-r_\varrho(\lambda) \approx E(\Xi_\lambda) + Sd(\Xi_\lambda) \left(z_1 + \frac{z_2 - 1}{6} Sk(\Xi_\lambda) \right)$$

with coefficients

$$\begin{aligned} z_1 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q) dq = -\frac{\phi(\Phi^{-1}(1-c))}{1-c} \\ z_2 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)^2 dq = 1 - \frac{\phi(\Phi^{-1}(1-c))}{1-c} \Phi^{-1}(1-c) \end{aligned}$$

depending on the confidence level $c < 1$ ¹.

Putting this together, we get a third expression for the index of satisfaction.

$$\begin{aligned} -r_\varrho(\lambda) &\approx \lambda p \left(\mu + \frac{1}{2}\mu^2 + \frac{1}{2}\Sigma \right) - |\lambda| p \frac{\phi(\Phi^{-1}(1-c))}{1-c} \sqrt{\Sigma} \\ &\quad \cdot \left(\sqrt{(1 + \mu)^2 + \frac{1}{2}\Sigma} + \frac{1}{2} \operatorname{sgn} \lambda \frac{(1 + \mu)^2 + \frac{1}{3}\Sigma}{(1 + \mu)^2 + \frac{1}{2}\Sigma} \Phi^{-1}(1-c) \sqrt{\Sigma} \right) \quad (3) \end{aligned}$$

This result agrees with (2) to lowest order in μ and Σ .

¹The trick to these integrals is to realize that $\phi'(z) = -z\phi(z)$.

4 Modeling Default

Our horizon asset value P is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$-L' = \lambda p (Y e^X - 1)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before², but now we add an independent default indicator $Y \sim \text{Bern}(1 - h)$ for default probability h .

²Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.