Financial Time Series

MFM Practitioner Module: Quantitiative Risk Management

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Outline

Invariants

We are generally working with financial timeseries data when calibrating models for the future value of financial variables such as the mark-to-market profit/loss on an asset holding.

- In some cases, such as equity shares, this may mean working with market prices (adjusted for dividends and splits).
- In other cases, such as for bonds or derivatives, it may mean working with derived quantities like yield or implied volatility.

Invariants

If we expect today that the meaning of a financial quantity of interest will remain uniform for the foreseeable future, we term it an invariant quantity. For example, the price or yield on a particular derivative or bond is *not* an invariant because the instrument will expire or mature on a known date.

Invariants

Indexes, Generics, & Synthetics

The challenge of identifying invariants for important classes of financial variables is addressed variously through indexes, synthetics, and generics.

- The S&P 500 equity index purports to represent the performance of typical large-cap U. S. listed equity securities.
- The Fed's CMT indexes purport to represent the performances of typical nominal U. S. Treasury bonds of particular tenors.
- Bloomberg futures generics represent the performances of the 1st, 2nd, etc. contract of a particular futures product.
- The CBOE VIX index purports to represent the performance of a delta-hedged position in one-month S&P 500 index options.

Investibility & Relevance of Invariants

If we intend to use an invariant as a proxy for an actual asset, it is important to think carefully about how the performance of the proxy can differ from the performance of the asset.

- The S&P 500 is an investible index whose performance can be replicated by an instantaneously fixed portfolio of equity shares, its performance is influenced by its dynamic composition and the dynamic correlation between constituents, which is obviously not relevant for individual equities.
- Other indexes, such as LIBOR (London interbank offered rate) or OIS (Federal Funds rate overnight index swap), are technically investible, but only by the treasury departments of banks; in particular they are *not* investible to broker-dealers.

Invariants

Seasonality

Some financial timeseries exhibit predictable patterns in time, or seasonality.

- A futures generic must roll whenever new contracts are issued. The actual profit/loss from rolling over a futures position is difficult to predict, and the generic makes no attempt at all.
  - For timeseries analysis purposes, you should omit roll dates from generics for your analysis.
- There may be predictable events, such as earnings announcements or the seasonal consumption patterns of certain commodities, that should be modeled as regimes.
  - This is a specialized topic in econometrics that we will not cover.

Innovations

We generally only care about the most recent level for a risk factor after our timeseries analysis is finished and we are looking at the loss distribution for a particular portfolio. For the analysis, we are more interested in the periodic innovations of the risk factor, such as the log-returns or simple differences.

- You can think of this as the difference operator applied to the index or its (natural) logarithm, $X_t \triangleq \nabla \log S_t$.

Drift

The conditional expected value $\mathrm{E}\left[\nabla \log S_t | \mathcal{F}_{t-1}\right]$ of the log of an index is termed the index drift μ_t.

Volatility

The conditional standard deviation $\sqrt{\operatorname{var}\left[\nabla \log S_t | \mathcal{F}_{t-1}\right]}$ is termed the index volatility σ_t.

Note that the drift and volatility are $\mathcal{F}_{t-1}$-measurable.

ARMA for Drift

White noise is a collection of i.i.d. r.v.'s ϵ_t with zero mean and finite variance σ_t^2. With $X_t = \mu_t + \varepsilon_t$ an innovation of an invariant, we call ε_t the residual.

Autoregressive Moving Average

An ARMA(p, q) process for the drift can be expressed as

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

for parameters ϕ_0, ϕ_1, ..., ϕ_p, θ_1, ..., θ_q.

- AR(1) is a simple model for mean reversion for the innovations around a long-run level $\phi_0/(1-\phi_1)$.
- The exponentially-weighted moving average EWMA can be represented by a version of ARMA:

$$\mu_t = \mu_{t-1} + \left(1 - 2^{-1/n_{\text{half-life}}}\right) \varepsilon_{t-1}$$

GARCH for Volatility

For financial data there is little to be gained in modeling drifts of timeseries data, because typically $|X_t| \gg \mu_t$.

- Furthermore, if X_t is a log-return, the drift probably ought to include a Jensen term like $-\frac{1}{2}\sigma_t^2$ which certainly does not fit into the ARMA form.

Generalized Autoregressive Conditional Heteroskedasticity

A GARCH(p, q) process for the conditional variance can be expressed as

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

for *non-negative* parameters $\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$.

GARCH for Volatility

Standardized Residuals

One application of GARCH models is to extract i.i.d. samples from timeseries. We define the standardized residuals as

$$Z_t \triangleq \frac{X_t - \mu_t}{\sigma_t}$$

To the extent that the GARCH model is correct, these are strict white noise.

GARCH(1,1)

By far the most common implementation of this model is GARCH(1,1). An important result about this model is that the unconditional variance is

$$\sigma^2 \triangleq \operatorname{var}[X_t] = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

as long as $\alpha_1 + \beta_1 < 1$.

Estimating GARCH

Let us continue to focus on GARCH(1,1). The principal technique for estimating the parameters of a GARCH process is maximum likelihood, but with several caveats:

- We do not know the marginal densities of the residuals and they are not identical
- We do not know know ε_0 or σ_0 (assume $t = 1$ is the first observed innovation)
- While we may assume that they are i.i.d., we may not know the exact density of the standardized residual $f_Z(\cdot)$

We address these through the quasi-MLE, in which we note that the multivariate density is the product of the conditional densities, and we assume that the residuals are normal:

$$\log f_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n|\varepsilon_0,\sigma_0}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n) = -\frac{1}{2}\sum_{t=1}^{n}\log\left(2\pi\sigma_t^2\right) + \frac{\varepsilon_t^2}{\sigma_t^2}$$

Estimating GARCH

Variance Targeting

Assuming that the unconditional variance of the innovations exists, it is advisable to set the intercept based on the sample variance.

$$\alpha_0 = \hat{\sigma}^2 (1 - \alpha_1 - \beta_1)$$

Then you are only using the QMLE to estimate α_1 and β_1. N.B.: You should probably put a lower bound on α_1 in this case, otherwise the β_1 could be degenerate.

Initialization

Assuming $t = 1$ is your first innovation, we need a way of determining σ_1^2 in terms of the parameters. That means you need to choose values for ε_0^2 and σ_0^2. One choice is to take both to be $\hat{\sigma}^2$. In combination with variance targeting, this means $\sigma_1^2 = \hat{\sigma}^2$.

Forecasting GARCH

In terms of forecasting, we already have σ_{n+1}^2. Say we are interested in $\mathrm{E}_n\left[\sigma_{n+2}^2\right]$ (the subscript on the expectation represents the filtration), we can write

$$\sigma_{n+2}^2 = \sigma^2\left(1-\alpha_1-\beta_1\right) + \sigma_{n+1}^2\left(\alpha_1 Z_{n+1}^2 + \beta_1\right)$$

so because $Z_{n+1} \sim \mathrm{SWN}(0,1)$

$$\mathrm{E}_n\left[\sigma_{n+2}^2\right] = \sigma_{n+1}^2\left(\alpha_1+\beta_1\right) + \sigma^2\left(1-\alpha_1-\beta_1\right)$$

Iterating this, we get the general result for integer $m > 0$,

$$\mathrm{E}_n\left[\sigma_{n+m}^2\right] = \sigma_{n+1}^2\left(\alpha_1+\beta_1\right)^{m-1} + \sigma^2\left(1-\left(\alpha_1+\beta_1\right)^{m-1}\right)$$

- The forecasts are a convex combination of the current conditional variance σ_{n+1}^2 and the unconditional, or long-run, variance σ^2.