# Multivariate Models

## MFM Practitioner Module: Quantitiative Risk Management

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January 20, 2016



Multivariate Random Variables
Spherical Random Variables
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# Multivariate Random Variables



We are going to pick up where we left off last term. The reading for this week (ch. 6) is long, but some of it should be review. In particular we have already seen most of the material in §6.1 on multivariate basics and §6.2 on variance mixtures of normals

- multivariate distribution and density concepts
- Maronna's M-estimator
- GIG-variance mixtures of normals (symmetric GH r.v.)
- affinity of conditional expectation with respect to condition event for multi-normals

# Spherical Random Variables

It is useful to build up a theory of multivariate random variables from geometric principles. By definition, a spherical random variable is distributionally invariant to rotations,

$$U\boldsymbol{X} \stackrel{d}{=} \boldsymbol{X}$$

where $U$ is a square matrix representation of a rotation, which means that $U'U = I$.

Spherical random variables have two equivalent defining properties,

$$\boldsymbol{a}'\boldsymbol{X} \stackrel{d}{=} \|\boldsymbol{a}\| X_1$$
$$\mathrm{E}\, e^{i\boldsymbol{t}'\boldsymbol{X}} = \psi\left(\boldsymbol{t}'\boldsymbol{t}\right)$$

for vectors $\boldsymbol{a}$ and $\boldsymbol{t}$. We term $\psi(\cdot)$ the characteristic generator of $\boldsymbol{X}$. We therefore write $\boldsymbol{X} \sim S_d(\psi)$ to denote a spherical random variable in $d$ dimensions with characteristic generator $\psi(\cdot)$.

# Elliptical Random Variables



An affine transformation of a spherical random variable is termed an elliptical random variable.

$$X \stackrel{d}{=} \boldsymbol{\mu} + A\boldsymbol{Y}$$

where $\boldsymbol{Y} \sim S_k(\psi)$ and $A$ is a $d \times k$ matrix.
The distributional invariance of $\boldsymbol{Y}$ to rotations means that $A$ is generally redundant. All we need to characterize $\boldsymbol{X}$ is $\boldsymbol{\mu}$, $\psi(\cdot)$, and $\Sigma = AA'$. But note that

$$E_d\left(\boldsymbol{\mu}, \Sigma, \psi(\cdot)\right) \stackrel{d}{=} E_d\left(\boldsymbol{\mu}, c\Sigma, \psi(\cdot/c)\right)$$

for $c > 0$, so $\Sigma$ may not necessarily be the covariance of $\boldsymbol{X}$.

- Note that $\Sigma$ need not be full rank. In this case, the rank of $\Sigma$ is at most $d \wedge k$.

# Elliptical Random Variables



## Some Properties

Say $\boldsymbol{X} \sim E_d\left(\boldsymbol{\mu}, \Sigma, \psi\right)$.

- linear combinations If $B$ $k \times d$ and $\boldsymbol{b}$ $k \times 1$ constants, then

$$B\boldsymbol{X} + \boldsymbol{b} \sim E_k\left(B\boldsymbol{\mu} + \boldsymbol{b}, B\Sigma B', \psi\right)$$

- if $\Sigma$ is full rank, then the non-negative scalar r.v.

$$R = \sqrt{(\boldsymbol{X} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X} - \boldsymbol{\mu})}$$

  is independent of $S = \Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})/R$ and $S$ is uniformly distributed on a unit sphere.

- convolutions If $\boldsymbol{Y} \sim E_d(\tilde{\boldsymbol{\mu}}, \Sigma, \tilde{\psi})$ independent of $\boldsymbol{X}$, then

$$\boldsymbol{X} + \boldsymbol{Y} \sim E_d\left(\boldsymbol{\mu} + \tilde{\boldsymbol{\mu}}, \Sigma, \psi \cdot \tilde{\psi}\right)$$

# Linear Factor Models



If $\boldsymbol{X}$ is a $d$-dim random variable, and we can write

$$\boldsymbol{X} = \boldsymbol{a} + B\boldsymbol{F} + \boldsymbol{\varepsilon}$$

where $\boldsymbol{F}$ is a $p$-dim random vector with $p < d$ and $\operatorname{cov} F > 0$, $B$ is a $d \times p$ matrix, the entries of $\boldsymbol{\varepsilon}$ are zero mean and uncorrelated, and $\operatorname{cov}(\boldsymbol{F}, \boldsymbol{\varepsilon}) = 0$, we call $\boldsymbol{F}$ the common factors and $B$ the factor loadings.

We would consider such a model or approximation if $d \gg p$.

Sometimes we have a idea about what the factors or loadings might be and they might even observable.

- In macroeconomic factor models, we observe the factors.
- In fundamental factor models, we observe the loadings.
- In statistical or latent factor models, we observe neither the factors nor the loadings.

## Capital Asset Pricing Model

CAPM for investments is an example of a macroeconomic factor model. It is typically applied to traded equity securities and a risk-free deposit as canonical "capital assets". We will take $\boldsymbol{X}$ to be the (simple) return on each risky capital asset over some investment period.

If $\boldsymbol{X}$ is normal and investors allocate to maximize expected exponential utility, then we can express the equilibrium solution as a single-factor model where $\boldsymbol{F}$ is the return on a broad index of risky capital assets.

The factor loadings $B_i$ can be determined by regression, and are often termed the asset "betas".

The intercept components turn out to be $a_i = r\tau\,(1 - B_i)$ where $r\tau$ is the return on the risk-free deposit.

## Fundamental Model

Sometimes it is useful to impose a classification scheme on the components of $\boldsymbol{X}$, for example an industry classification scheme or a geographic or demographic scheme. In this case, we generally know the non-zero loadings in $B$, but we do not observe the factors $\boldsymbol{F}$.

In this case, we can estimate timeseries for $\boldsymbol{F}$ in terms of timeseries for $\boldsymbol{X}$ according to ordinary least squares regression

$$\hat{\boldsymbol{F}}_t^{\mathrm{OLS}} = \left(B'B\right)^{-1} B'\boldsymbol{X}_t$$

if the variance of the residuals is the same (homoscedastic) or generalized least squares regression

$$\hat{\boldsymbol{F}}_t^{\mathrm{GLS}} = \left(B'\boldsymbol{\Upsilon}^{-1}B\right)^{-1} B'\boldsymbol{\Upsilon}^{-1}\boldsymbol{X}_t$$

if not.

# Principal Components



Principal components analysis is inspired by the concept of a statistical factor model, but since it is entirely endogenous it is better understood as a separate concept.

A covariance or correlation matrix $\Sigma$ has the property of being positive semi-definite, which means that $\boldsymbol{x}'\Sigma\boldsymbol{x} \geq 0$ for all compatible vectors $\boldsymbol{x}$. Therefore, by the spectral decomposition theorem, we can write

$$\Sigma = \Gamma\Lambda\Gamma'$$

where $\Lambda$ is a diagonal matrix with non-negative entries (the eigenvalues) and $\Gamma$ is a square matrix whose columns (the eigenvectors) are orthonormal, which means $\Gamma\Gamma' = I$.

# Principal Component Analysis



If $\Sigma$ has full rank $d$, all of the eigenvalues will be positive. The potential for dimension reduction comes from partitioning the model into the largest $k < d$ eignevalues and corresponding eigenvectors and relegating the remaining $d - k$ to the residual.

## Principal Components as Factors

Let $d \times 1$ $\boldsymbol{Y} = \Gamma'(\boldsymbol{X} - \boldsymbol{\mu})$ where $\boldsymbol{\mu}$ is the mean of $\boldsymbol{X}$. Partition $\boldsymbol{Y}$ and $\Gamma$ into $k \times 1$ $\boldsymbol{Y}_1$ and $(d-k) \times 1$ $\boldsymbol{Y}_2$ and $d \times k$ $\Gamma_1$ and $d \times (d-k)$ $\Gamma_2$ and let $\varepsilon = \Gamma_2 \boldsymbol{Y}_2$, then

$$X = \boldsymbol{\mu} + \Gamma_1 \boldsymbol{Y}_1 + \varepsilon$$

and $\varepsilon$ *almost* satisfies the assumptions for a linear factor model.