Copulas and Dependence

MFM Practitioner Module: Quantitiative Risk Management

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Introduction

We have discussed continuous multivariate random variables and some broad parametric classes. It is clear that each instance of these involves at least one class of univariate random variable in the form of the marginals for the components. But it is also clear that the characterization of the original multivariate r.v. is not simply a collection of these marginal characterizations. There is a structure, with its own parameters, that connects them together.

- This is the copula.

For me, this is the prototypical example; and multivariate random variables are a rich source for parametric copulas. But it is not the only source. In fact, any random variable whose sample space is a unit hypercube with standard uniform margins is a copula.

Sklar's Theorem

To the extent that the *joint* density is not just a product of the *marginal* densities, there is dependence.

Factorization

This ratio can be expressed as

$$f_U\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots\right) \triangleq \frac{f_{(X_1,X_2,\ldots)}(x_1, x_2, \ldots)}{f_{X_1}(x_1) f_{X_2}(x_2) \cdots}$$

Copula

Sklar's theorem says this is always possible. More generally, $f_U : [0,1]^d \mapsto \mathbb{R}^+$ is a density function that characterizes a new random variable, U, that encapsulates the dependence structure of X.

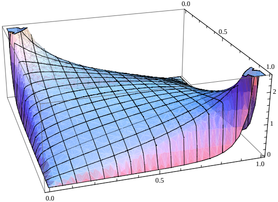
Note that independence means $f_U \equiv 1$

Copulas

Normal (Gaussian) Copula

When dependence can be entirely described by correlation, the Gaussian copula can be appropriate. For $d = 2$,

$$f_U(u) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-\rho}{1-\rho^2}\left(\rho \operatorname{erfc}^{-1}(2u_1)^2 \cdots \right.\right.$$
$$\left.\left. - 2\operatorname{erfc}^{-1}(2u_1)\operatorname{erfc}^{-1}(2u_2) + \rho \operatorname{erfc}^{-1}(2u_2)^2\right)\right]$$



Gaussian copula density for $\rho = \frac{1}{2}$

Tail Dependence

Upper & Lower Tail Dependence

Tail dependence is a pair-wise measure of the concordance of extreme outcomes.

$$\lambda_U = \lim_{p \uparrow 1} \mathrm{P}\left\{X > F_X^{\leftarrow}(p) | Y > F_Y^{\leftarrow}(p)\right\}$$

$$\lambda_L = \lim_{p \downarrow 0} \mathrm{P}\left\{X \leq F_X^{\leftarrow}(p) | Y \leq F_Y^{\leftarrow}(p)\right\}$$

The normal copula fails to exhibit tail dependence: extreme outcomes are essentially independent.

This is a problem, because in practice an extreme outcome in one dimension often acts to cause extreme outcomes in other dimensions. Developing practical alternatives that include this contagion effect is an active area of research.

Measures of Concordance

Several measures of concordance have been developed. Their definitions are motivated by the properties of their estimators, which we will not discuss just yet. Each ranges from -1 to 1, with 0 for independence. In order of generality, we have

1. Pearson's rho. This is the classical linear correlation measure.
2. Spearman's rho. This is correlation applied to the grades, $F_X(X)$. It is a simple measure of dependence that is not sensitive to margins.
3. Kendall's tau. This is a pure copula measure. It is based on rank-order correlations.

N.B.: While independence implies zero concordance (under any of these definitions), zero concordance does not imply independence.

Kendall's tau

Kendall's tau can be defined as

$$\tau = 4\,\mathrm{E}\,F_U(U_1, U_2) - 1$$

where F_U is the distribution function characterizing the copula of X. It is the probability of concordance minus the probability of discordance for two independent draws of X.

Relationship with other measures

In general Spearman's rho is bounded by

$$\frac{3|\tau| - 1}{2}\operatorname{sgn}\tau \quad \& \quad \frac{1 + 2|\tau| - \tau^2}{2}\operatorname{sgn}\tau$$

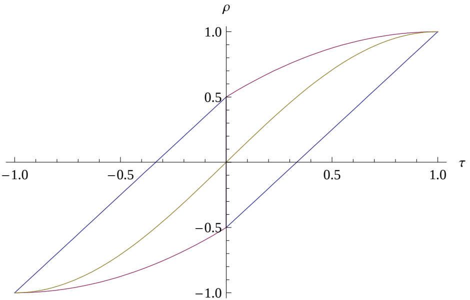
For a Gaussian copula, Pearson's rho is

$$\rho = \sin\left(\tfrac{\pi}{2}\tau\right)$$

One can use this to define the pseudo-correlation.

Kendall's tau

The relationship between Kendall's tau and Spearman's and Pearson's rho is illustrated by this graph.



For a given level of Kendall's tau, Spearman's rho is bounded by the two outer curves. Pearson's rho for a Gaussian copula is the curve through the origin.

Normal Mixture Copulas

A normal mixture copula is simply the copula from a normal mixture multivariate random variable. The elliptical copula is an important subclass.

Elliptical Copula

An elliptical random variable is described by a mean vector, a dispersion matrix, and a characteristic generator function. It should be clear that the mean vector has no role in the copula. It should also be clear that the diagonal entries of the dispersion matrix also do not play a role.

Generally, an elliptical copula is parameterized by a semi-definite matrix with unit diagonals, which describe pair-wise dependence, and one or several shape parameters related to the characteristic generator.
The Gaussian copula is an example. Another important example is the t_ν copula, which we will work with in this week's exercise.

Archimedean Copulas

There are on the order of $d^2/2$ parameters to estimate for an elliptical copula. If you are dealing with a very large dimension, such as in a retail or securitization context, you either need a factor model to reduce the dimension or you should consider an Archimedean copula.

Archimedean Copulas

An Archimedean copula is defined in terms of a generator, a decreasing continuous function $\psi : [0, \infty) \mapsto [0, 1]$ with $\psi(0) = 1$ and $\lim_{t\to\infty} \psi(t) = 0$. The copula distribution is

$$C(u_1, u_2, \ldots, u_d) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots + \psi^{-1}(u_d)\right)$$

Three common single-parameter examples are the Gumbel, Clayton, Frank.