Aggregate Risk
MFM Practitioner Module: Quantitative Risk Management

John Dodson

February 3, 2016
Introduction

As we discussed last semester, the general goal of risk measurement is to come up with a single metric that can be used to make financial risk management decisions. This is a difficult task; but axiomatic concepts such as –

- monotonicity
- translation-invariance
- law-invariance
- sub-addititivity
- quasi-convexity
- coherence
- positive homogeneity

have proven to be useful in at least defining the problem. The notions of acceptance sets and the dual representation have also led to important insights.
Acceptance Sets

An important observation about coherent risk measures is that they can always be expressed in terms of convex sets $A_\varrho$ of acceptable loss outcomes.

$$\varrho(L) = \inf \{ m \in \mathbb{R} : L - m \in A_\varrho \}$$

Clearly if $\varrho$ is law-invariant, then $A_\varrho$ depends on the risk measure $\mathbb{P}$.

**Example: Value-at-Risk**

Value-at-risk is a good example. It is monotone and translation-invariant, but not necessarily coherent. The acceptable set the set of losses such that $\mathbb{P}(L > 0) \leq 1 - \alpha$. Depending on $\mathbb{P}$, this may or may not be convex.

Insights from the theory of acceptance sets was important in the development of the principle of coherence.
Dual Representation

The notion of the dual representation of risk measures is an important theoretical insight which has subsequently influenced practical developments in risk measurement. It is similar to the notion of acceptance sets of losses, but instead we consider an acceptable set of probability measures.

Dual Representation for Coherent Risk Measures

If \( \varrho(L) \) is coherent, there is a set of probability measures \( Q_{\varrho} \) for which

\[
\varrho(L) = \max \left( \mathbb{E}^Q L : Q \in Q_{\varrho} \right)
\]

Example: Expected Shortfall

For expected shortfall, which is generally coherent, we have

\[
Q_{\varrho} = \left\{ Q : \frac{dQ}{dP} \leq \frac{1}{1 - \alpha} \right\}
\]

in terms of the Radon-Nikodym derivative.
Expected shortfall is the canonical example of a so-called distortion risk measure, which are defined in general as

$$\varrho(L) = \int_0^1 q_u(L) \, dD(u)$$

in terms of the loss quantile function $q_u(L) \triangleq F_L^{-1}(u)$ and a convex, increasing, absolutely continuous distortion function $D : [0,1] \mapsto [0,1]$ with $D(0) = 0$ and $D(1) = 1$. Distortion risk measures are coherent. They are also comonotone additive, which is a feature from last week.

**Example: Expected Shortfall**

For expected shortfall, the distortion function is just

$$D(u) = (1 - \alpha)^{-1}(u - \alpha)^+$$

All distortion risk measures can be expressed as an expectation of $ES_\alpha$ under some measure $\mu(\alpha)$. 

Coherent Stress Tests

For a positive homogeneous risk measure and affine loss, we can write

\[ \varrho(m + \lambda'X) = m + r_\varrho(\lambda) \]

If we define the set of risk factor scenarios \( S_{\varrho} \) as

\[ S_{\varrho} = \left\{ x \in \mathbb{R}^d : u'x \leq r_\varrho(u) \quad \forall u \in \mathbb{R}^d \right\} \]

then

\[ \varrho(L) = \sup \left\{ m + \lambda'x : x \in S_{\varrho} \right\} \]

Note that the scenario set does not depend on the allocations \( \lambda \).

Since this is equivalent to a dual representation with \( Q_\varrho \) consisting of degenerate measures on the elements of \( S_{\varrho} \), it is coherent.
Linear Loss and Elliptical Risk Factors

Suppose that the risk factors are an elliptic random vector, \( \mathbf{X} \sim E_d (\mu, \Sigma, \psi) \), and suppose that potential losses are affine in the risk factors, \( L = m + \lambda' \mathbf{X} \). Then for any law-invariant, translation-invariant, positive homogeneous risk measure \( \varrho \),

\[
\varrho(L) = m + \lambda' \mu + \varrho(Y_1) \sqrt{\lambda' \Sigma \lambda}
\]

where \( Y \sim S_d(\psi) \).

If furthermore \( \varrho(Y_1) > 0 \) and \( \mathbf{X} \) has a finite covariance matrix, then \( \varrho \) is a sub-additive risk measure, and

\[
\varrho(L) = \mathbb{E} L + k_\varrho \sqrt{\text{var} L}
\]

which means that \( \varrho \) is consistent with weak stochastic dominance and that optimal portfolios lie on the Markowitz mean-variance efficient frontier.
Euler Decomposition

A notable feature of the standard deviation of a weighted sum of correlated random variables is that it can be expressed as a weighted sum of partial standard deviations.

\[ \varrho(L) = \varrho(\lambda' L) = r_\varrho(\lambda) \triangleq \sqrt{\lambda' \Sigma \lambda} = \lambda' \frac{\sum \lambda}{r_\varrho(\lambda)} \]

This is true of any risk measures that are (first-order) positive homogeneous, which is to say any risk measure such that for \( \forall \lambda \geq 0, \varrho(\lambda L) = \lambda \varrho(L) \), which is clearly true for standard deviation.

**Allocation by Gradient**

In the case where \( \lambda = 1 \) hence \( L = L_1 + L_2 + \cdots + L_d \), this means

\[ \varrho(L) = \sum_{i=1}^{d} \frac{\partial r_\varrho}{\partial \lambda_i}(1) \triangleq \sum_{i=1}^{d} AC_i^\varrho \]