Quantitative Risk Management
Spring Assignment

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This assignment is not a regular homework. It is worth half of the module grade for the spring term. Please share your solution with me through Google Drive before the beginning of the classroom session on Wednesday, March 9 (extended from March 2).

Please write up your results into a report and submit a PDF version (My edition of LibreOffice can recognize most word processing formats, but often the formatting, equations, and exhibits are altered or corrupted.) You may be able to covert directly to PDF from your word processor or type setter, or you may be able to "print to PDF" using a print driver.

I expect you to work independently and to cite your sources. Please remind yourself of the University's definition of scholastic dishonesty (see the syllabus).

Valuing a CDS under a structural model

In this problem, you will explore credit default swap valuation in the solvency version of the structural model, which was originally contemplated in the same Merton (1974) paper that is cited in our text, reintroduced in Leland (1994) and extended in Leland & Toft (1996) and again in Hilberink & Rogers (2002).

You worked with the solvency version of the structural model last semester in the context of equity valuation. Here we set-up the model again using the notation of §10.3 from the text.

A firm has assets and liabilities. The assets are not liquid and cannot be used to pay interest on the liabilities; rather, the assets generate earnings which are used to pay interest to creditors, taxes to taxing authorities, and dividends to owners. Retained earnings are immediately reinvested into (illiquid) assets[1].

The value of the assets is a stochastic process (V_t), which is a geometric Brownian motion with (constant) volatility σ. The (constant) after-tax earnings rate on assets is δ. Since an investor could own shares in both the debt and the equity, s/he could construct a position in the asset process. Therefore, there should be a unique risk-neutral measure such that the process for the assets is

$$dV_t = (r - \delta)V_t\,dt + \sigma V_t\,dW_t \tag{1}$$

where r is the (constant) risk-free interest rate and (W_t) is a standard Brownian motion.

The (fixed) interest burden is rK. In this model, creditors are effectively short a perpetual American-style put, which the owners (or more realistically the bankruptcy trustees) can exercise to give the remaining

[1] Allowing some unobservable fraction of the assets to remain liquid is a natural mechanism for introducing a "fuzzy boundary" to solve the accessible default problem in this model, but it complicates the analysis.

assets of the firm to the creditors in exchange for cancelling the debt contracts. The strike price for this put is K. The value is

$$p_t = (K - V_t \wedge L) \left(\frac{L}{V_t \vee L} \right)^{\gamma}$$

where the (dimensionless) volatility scale parameter is

$$\gamma = \frac{r - \delta}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}$$

and the (Markovian) exercise event is defined by $\inf \{t : V_t \leq L\}$ where

$$L = \frac{K}{1 + \frac{1}{\gamma}}$$

The value processes for the debt and equity are $B_t = K - p_t$ and $S_t = V_t - K + p_t$. Discounted at the risk-free rate and accounting for cashflows, (V_t), (B_t), and (S_t) are all non-negative martingales under the risk-neutral measure.

The credit event for a credit default swap is not the exercise of the put. That is too late. Rather, it is the first moment when the the earnings are insufficient to meet the interest burden ("insolvency"). The default stopping time τ is therefore defined by

$$\tau = \inf \left\{ t : V_t \leq L' \right\} \tag{2}$$

where

$$L' = \frac{rK}{\delta} = \frac{L}{1 - \gamma \frac{\sigma^2}{2r}}$$

In order to value credit default swaps, we need the (risk-neutral) probability distribution of the default stopping time τ. This involves the distribution of the first passage time of a drifted Brownian motion, which is a standard result from the theory of stochastic processes.

The first passage time of a drifted Brownian motion

If τ is the first passage time for a drifted Brownian motion,

$$\tau = \inf \{ t : \theta t + W_t \leq M \} \tag{3}$$

with $M < 0$, then it is an inverse Gaussian random variable with probability density and distribution function

$$f_\tau(t) = \frac{-M}{\sqrt{2\pi t^3}} e^{-(\theta t - M)^2/(2t)} \tag{4}$$

$$F_\tau(t) = e^{2M\theta} \Phi \left(\frac{M}{\sqrt{t}} + \theta \sqrt{t} \right) + \Phi \left(\frac{M}{\sqrt{t}} - \theta \sqrt{t} \right) \tag{5}$$

Note that if $\theta > 0$, $\lim_{t \to \infty} F_\tau(t) = e^{2M\theta} < 1$, so there is a positive probability associated with the event $\{\tau = \infty\}$.

Problems

1. Solve the stochastic differential equation (1) for V_t in terms of V_0. From this determine the values for the generic drift θ and threshold M in (3) that correspond to the definition of default in (2). **(30 points)**

2. The payoff of a credit default swap is $1 - B_\tau/B_0$. A feature of this model is that this is an $\mathcal{F}_0$-measurable quantity. Evaluate it in terms of the model parameters and V_0 or M. **(10 points)**

3. Determine the swap rate $s_0(T)$ such that the risk-neutral expected value of the risk-free discounted value of the (continuous) stream of premium payments until $\tau \wedge T$ equals that of the default-contingent payoff[2]. **(20 points)**

4. Numerically evaluate $s_0(10)$, the ten-year CDS rate, for the "bank" that was defined in our first fall exercise. **(10 points)**

Grading Rubric

Thirty out of one hundred points will be based on the following criteria:

- You follow the instructions. **(10 points)**
- You include adequate citations. **(10 points)**
- Your write-up is clear and professional. **(10 points)**

Solution

Inverse Gaussian parameters

The solution for V_t is achieved in the usual fashion by taking logs and applying Itô's Lemma,

$$d \log V_t = \left(r - \delta - \tfrac{1}{2}\sigma^2\right) dt + \sigma dW_t \tag{6}$$

hence

$$V_t = V_0 e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W_t} \tag{7}$$

The default condition is

$$V_0 e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W_t} \leq L' \tag{8}$$

or

$$\frac{r-\delta-\frac{1}{2}\sigma^2}{\sigma} t + W_t \leq \frac{1}{\sigma} \log \frac{L'}{V_0} \tag{9}$$

Comparing this to (3), we see that

$$\boxed{\theta = \frac{r-\delta}{\sigma} - \frac{\sigma}{2}} \tag{10}$$

[2]Express the solution in terms of the Gauss error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ or the normal distribution function $\Phi(x) = \int_{-\infty}^x \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(x/\sqrt{2}\right)$.

and

$$\boxed{M_0 = \frac{1}{\sigma} \log \frac{L'}{V_0}} \tag{11}$$

The "distance to default" M_0 is always negative. The subscript is to remind us that this is a stochastic quantity.

Note that θ can be simplified in terms of γ,

$$\theta = \frac{\gamma\sigma}{2} - \frac{r}{\gamma\sigma} \tag{12}$$

and in turn we can solve for γ in terms of θ,

$$\gamma\sigma = \theta + \sqrt{\theta^2 + 2r} \tag{13}$$

which will be convenient below.

The drift θ might be positive or negative depending on the relative magnitudes of r and $\frac{1}{2}\gamma^2\sigma^2$.

Credit default swap payoff

At default, the value of the debt is B_τ is $K - p_\tau$. Since $V_\tau = L'$, this evaluates to

$$B_\tau = K - (K - L' \wedge L)\left(\frac{L}{L' \vee L}\right)^\gamma \tag{14}$$

As long as $r > 0$ then $\gamma > 0$ and $0 < L < L'$. So

$$B_\tau = K - (K - L)\left(\frac{L}{L'}\right)^\gamma \tag{15}$$

for $\tau > 0$ (prior to default),

$$B_0 = K - (K - L)\left(\frac{L}{V_0}\right)^\gamma \tag{16}$$

We can get that the CDS payoff from these,

$$\boxed{1 - \frac{B_\tau}{B_0} = \frac{e^{-\gamma\sigma M_0} - 1}{(1+\gamma)\left(\frac{V_0}{L}\right)^\gamma - 1}} \tag{17}$$

which here has L' eliminated through the introduction of (11).

Note that the payoff is technically undefined if $\tau = \infty$, which we see is a possibility if $\theta > 0$. We can define it arbitrarily to be (17) in this case as long as we never attempt to use the payoff at infinity for any calculations[3].

[3]This was pointed out by a student. You know who you are!

Credit default swap rate

In order to value the CDS, we need two expectations of functions of the default stopping time:

$$\mathrm{E}_0\left[\chi_{\{\tau<T\}}\right] = \mathrm{P}_0\left\{\tau < T\right\} = \int_0^T f_\tau(t)\, dt \tag{18}$$

and

$$\mathrm{E}_0\left[e^{-r\tau}\chi_{\{\tau<T\}}\right] = \int_0^T e^{-rt} f_\tau(t)\, dt \tag{19}$$

These integrals are actually quite similar once you recognize that, by completing the square in the exponential of (4),

$$e^{-rt} f_\tau(t;\theta,M) = e^{M\left(\theta+\sqrt{\theta^2+2r}\right)} f_{\tau'}(t;-\sqrt{\theta^2+2r},M) \tag{20}$$

The value the floating default-contingent insurance leg less the value of the fixed premium leg should be zero at inception when the swap rate $s_0(T)$ is set.

$$0 = \mathrm{E}_0\left[\chi_{\{\tau<T\}} e^{-r\tau}\left(1-\frac{B_\tau}{B_0}\right) - \int_0^{\tau\wedge T} e^{-rt} s_0(T)\, dt\right] \tag{21}$$

Evaluating the integral,

$$0 = \mathrm{E}_0\left[\chi_{\{\tau<T\}} e^{-r\tau}\left(1-\frac{B_\tau}{B_0}\right) - \frac{s_0(T)}{r}\left(1-\chi_{\{\tau<T\}}e^{-r\tau} - \left(1-\chi_{\{\tau<T\}}\right)e^{-rT}\right)\right] \tag{22}$$

combining terms,

$$0 = -\frac{s_0(T)}{r}\left(1-e^{-rT}\right) - \frac{s_0(T)}{r}e^{-rT}\,\mathrm{E}_0\left[\chi_{\{\tau<T\}}\right] + \left(\frac{s_0(T)}{r} + 1 - \frac{B_\tau}{B_0}\right)\mathrm{E}_0\left[\chi_{\{\tau<T\}}e^{-r\tau}\right] \tag{23}$$

and applying (20), (13), and (17), we get

$$0 = -\frac{s_0(T)}{r}\left(1-e^{-rT}\right) - \frac{s_0(T)}{r}e^{-rT} F_\tau(T) + \left(\frac{s_0(T)}{r} + \frac{e^{-\gamma\sigma M_0}-1}{(1+\gamma)\left(\frac{V_0}{L}\right)^\gamma - 1}\right) e^{\gamma\sigma M_0} F_{\tau'}(T) \tag{24}$$

which can be solved for the swap rate

$$\boxed{s_0(T) = \frac{\frac{r}{(1+\gamma)\left(\frac{V_0}{L}\right)^\gamma - 1}}{1 + \frac{(1-F_{\tau'}(T)) - e^{-rT}(1-F_\tau(T))}{\left(1-e^{\gamma\sigma M_0}\right)F_{\tau'}(T)}}} \tag{25}$$

where

$$F_\tau(T) = e^{2M_0\theta}\Phi\left(\frac{M_0}{\sqrt{T}} + \theta\sqrt{T}\right) + \Phi\left(\frac{M_0}{\sqrt{T}} - \theta\sqrt{T}\right) \tag{26}$$

$$F_{\tau'}(T) = e^{2M_0(\theta-\gamma\sigma)}\Phi\left(\frac{M_0}{\sqrt{T}} + (\theta-\gamma\sigma)\sqrt{T}\right) + \Phi\left(\frac{M_0}{\sqrt{T}} - (\theta-\gamma\sigma)\sqrt{T}\right) \tag{27}$$

Note that since $\theta - \gamma\sigma < 0$, $\lim_{T\to\infty} F_{\tau'}(T) = 1$, so

$$\lim_{T\to\infty} s_0(T) = \frac{r}{(1+\gamma)\left(\frac{V_0}{L}\right)^\gamma - 1} \tag{28}$$

On the other hand, since $F_{\tau'}(0) = 0$,

$$s_0(0) = 0$$

which is consistent with the accessible nature of the default stopping time in this model.

Worked example

Let's examine the set-up from the September 9, 2015, assignment. The firm has a current asset value of $V_0 = 100$ (million USD). The after-tax income rate is $\delta = 0.048$ (per year) and the (annual) volatility rate is $\sigma = 0.02$. With the (annual) risk-free interest rate $r = 0.01$, we determined that $\gamma \approx 0.26$.

The fixed expenses are $Kr = 3.04$ (million USD per year), so the liquidation strike price is $K = 304$ (million USD), the liquidation threshold is $L = K/(1 + \frac{1}{\gamma}) \approx 63.0$ (million USD), and the insolvency threshold of $L' = rK/\delta \approx 63.3$ (million USD) is almost identical.

The (annual) distance to default, according to (11) is $M_0 \approx -22.8$ and the (annual) drift according to (12) is $\theta \approx -1.9$.

According to (28) the asymptotic credit spread is $s_0(\infty) \approx 0.0236$ (per year) or 236 basis points.

At the ten-year tenor, $T = 10$ (years), the (risk-neutral) default probability is $F_\tau(10) \approx 0.133$. The adjusted probability is $F_{\tau'}(10) \approx 0.137$ and the credit default swap rate according to (25) is $\boxed{s_0(10) \approx 0.0039}$ (per year) or 39 basis points.

While this result might seem reasonable, it is also a little deceptive since the credit default rate for any tenor less than than about 7.8 years is under one basis point. This is the accessibility effect. Notably, it is not clear how the Duffie-Lando "fuzzy boundary" fix would help us here, since the insolvency threshold and the liquidation threshold are so close together. Perhaps we need to introduce a jump process into the asset value.

Appendix: Inverse Gaussian distribution function

An approach to the integral in (18),

$$\int_0^T \frac{-M}{\sqrt{2\pi t^3}} e^{-(\theta t-M)^2/(2t)}\,dt$$

based on [1], is to change variables to $u = (M - \theta t)/\sqrt{t}$. This is monotonic over the integration region as long as $\theta < 0$; and since we can use (20) with $r = 0$ otherwise, let's assume that $\theta < 0$ at this stage and go back and check the non-negative case later.

$$u = \frac{M}{\sqrt{t}} - \theta\sqrt{t} \tag{29}$$

$$u(0) = -\infty \qquad u(T) = \frac{M}{\sqrt{T}} - \theta\sqrt{T}$$

$$du = \frac{-M}{\sqrt{t^3}}\left(\frac{1}{2} + \frac{\theta t}{2M}\right)\,dt \tag{30}$$

We can invert (29) to get

$$t = \frac{M}{\theta} + \frac{u}{2\theta^2}\left(u + \sqrt{u^2 + 4M\theta}\right) \tag{31}$$

and with some manipulation we can re-write (30) as

$$\left(1 - \frac{u}{\sqrt{u^2+4M\theta}}\right)\,du = \frac{-M}{\sqrt{t^3}}\,dt \tag{32}$$

so

$$\begin{aligned}\int_0^T \frac{-M}{\sqrt{2\pi t^3}} e^{-(\theta t-M)^2/(2t)}\,dt &= \int_{-\infty}^{\frac{M}{\sqrt{T}}-\theta\sqrt{T}} \frac{1}{\sqrt{2\pi}}\left(1 - \frac{u}{\sqrt{u^2+4M\theta}}\right)e^{-\frac{1}{2}u^2}\,du \\ &= \Phi\left(\frac{M}{\sqrt{T}} - \theta\sqrt{T}\right) - \int_{-\infty}^{\frac{M}{\sqrt{T}}-\theta\sqrt{T}} \frac{1}{\sqrt{2\pi}}\frac{u}{\sqrt{u^2+4M\theta}}e^{-\frac{1}{2}u^2}\,du\end{aligned}$$

The integral in the second term above is facilitated by a further change of variables, $v = -\sqrt{u^2 + 4M\theta}$, whereby

$$dv = -\frac{u}{\sqrt{u^2+4M\theta}}\,du$$

The lower integration bound remains $-\infty$, while the upper bound becomes $-\left|\frac{M}{\sqrt{T}} + \theta\sqrt{T}\right|$, and since both M and θ are negative, we get

$$\int_{-\infty}^{\frac{M}{\sqrt{T}}-\theta\sqrt{T}} \frac{1}{\sqrt{2\pi}}\frac{u}{\sqrt{u^2+4M\theta}}e^{-\frac{1}{2}u^2}\,du = -\int_{-\infty}^{\frac{M}{\sqrt{T}}+\theta\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2+2M\theta}\,dv$$

Putting this together, we get the result

$$\int_0^T \frac{-M}{\sqrt{2\pi t^3}} e^{-(\theta t-M)^2/(2t)}\,dt = e^{2M\theta}\Phi\left(\frac{M}{\sqrt{T}} + \theta\sqrt{T}\right) + \Phi\left(\frac{M}{\sqrt{T}} - \theta\sqrt{T}\right) \tag{33}$$

Now let's check the case $\theta > 0$. From (20), $f_\tau(t;\theta,M) = e^{2M\theta} f_{\tau'}(t;-\theta,M)$, so all we need to do is replace $\theta \to -\theta$ in (33) and scale the result by $e^{2M\theta}$. But (33) is unchanged by this.

Finally, let's check $\theta = 0$, which is a special case because (31) as it is written is undefined. In this case, (32) becomes

$$2\ du = \frac{-M}{\sqrt{t^3}}\ dt \tag{32'}$$

and

$$\int_0^T \frac{-M}{\sqrt{2\pi t^3}} e^{-M^2/(2t)}\ dt = 2\int_{-\infty}^{\frac{M}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\ du = 2\Phi\left(\frac{M}{\sqrt{T}}\right)$$

But this is also consistent with (33). So in conclusion the result for the distribution function holds for all (real) θ.

References

[1] R. S. Chhikara and J. L. Folks. Estimation of the inverse gaussian distribution function. *Journal of the American Statistical Association*, 69(345):250–254, March 1974.

[2] D. Duffie and D. Lando. Term structure of credit spreads with incomplete accounting information. *Econometrica*, 69(3):633–664, May 2001.

[3] B. Hilberink and L. C. G. Rogers. Optimal capital structure and endogenous default. *Finance and Stochastics*, 6(2):237–263, April 2002.

[4] H. E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49(1):157–196, March 1994.

[5] H. E. Leland and K. Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, 51(3):987–1019, July 1996.

[6] R. C. Merton. On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29(2):449–470, May 1974.