Quantitative Risk Management
Case for Week 6

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Background

The identification of the non-degenerate normalized n-block maxima with the Fréchet, Gumbel, or Weibull random variables is the collective product of a number of probabilists working in Europe from the 1920's to the 1940's. For a survey of the important proofs, see [2].

This was a foundational result that helped define statistics and probability as a distinct sub-field of mathematics.

Today it is referred to as the "first extreme value theorem", because an extension was discovered in the 1970's that connected it with, and provided a theoretical foundation for, a panoply of contemporaneous empirical "power-law" results. This is now termed the "second extreme value theorem".

The Second Extreme Value Theorem

The second theorem says, in brief, that the relative probabilities of sufficiently extreme events are arbitrarily similar to the relative probabilities one would assign to a generalized Pareto random variable.

I would like to give an informal demonstration of this connection here. Please see [1] for a more careful treatment.

Let's start with the excess distribution. If a random variable X has distribution $F(\cdot)$, the "excess distribution" is

$$\begin{aligned} F_\eta(x) &\triangleq \mathrm{P}\left[X-\eta \le x | X > \eta\right] \\ &= \frac{F(x+\eta)-F(\eta)}{1-F(\eta)} \\ &= 1-\frac{1-F(x+\eta)}{1-F(\eta)} \end{aligned}$$

Let's further assume that the normalized n-block maxima converges to a Fréchet (other cases are similar). This means that for some sufficiently large n,

$$F^n\left(c_n x + b_n\right) \approx \exp\left(-(1+\xi x)^{-1/\xi}\right)$$

or

$$
\begin{aligned}
F(x) &\approx \exp\left(-\frac{1}{n}\left(1+\xi\frac{x-b_n}{c_n}\right)^{-1/\xi}\right) \\
&\approx 1-\frac{1}{n}\left(1+\xi\frac{x-b_n}{c_n}\right)^{-1/\xi}
\end{aligned}
$$

where we are using the approximation $e^{-x} \approx 1-x$ for small x in the second line.

Substituting this into the excess distribution, we get

$$
\begin{aligned}
F_\eta(x) &\approx 1-\frac{\left(1+\xi\frac{x+\eta-b_n}{c_n}\right)^{-1/\xi}}{\left(1+\xi\frac{\eta-b_n}{c_n}\right)^{-1/\xi}} \\
&= 1-\left(1+\xi\frac{x}{c_n+\xi(\eta-b_n)}\right)^{-1/\xi}
\end{aligned}
$$

So, defining $\beta(\eta) \triangleq c_n+\xi(\eta-b_n)$, we get the result

$$
F_\eta(x) \approx 1-\left(1+\xi\frac{x}{\beta(\eta)}\right)^{-1/\xi}
$$

which says that the excess is approximated by a generalized Pareto.

References

[1] August A. Balkema and Laurens de Haan. Residual life time at great age. *The Annals of Probability*, 2:792–804, 1974.

[2] Boris V. Gnedenko. On the limit distribution of the maximum term of a random series. (in French) *Annals of Mathematics*, 44:423–453, 1943.