Let us consider the expected shortfall index of satisfaction for a very simple portfolio: \( \lambda \) shares in an asset whose value today is \( p > 0 \) and whose horizon value \( P \) is lognormal.

Let us assume that the objective measure is mark-to-market profit; therefore in the text’s notation, we have (apologies for the signs)

\[
-L = \lambda (P - p) = \lambda p (e^X - 1)
\]

where the invariant total return is normal \( X \sim \mathcal{N}(\mu, \Sigma) \) with mean \( \mu \) and variance \( \Sigma > 0 \). The risk measure is

\[
-\varrho(L) = -r_{\varrho}(\lambda) = \frac{1}{1-c} \int_0^{1-c} F_{-L}(q) \, dq
\]

for confidence level \( c < 1 \) in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

\[
F_{-L}(z) = P \{ -L < z \} = P \{ \lambda p (e^X - 1) < z \} = P \left\{ X \text{ sgn } \lambda < \log \left( 1 + \frac{z}{\lambda p} \right) \text{ sgn } \lambda \right\} = P \left\{ \frac{X - \mu}{\sqrt{\Sigma}} \text{ sgn } \lambda < \frac{\log \left( 1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma}} \text{ sgn } \lambda \right\} = \Phi \left( \frac{\log \left( 1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma} \text{ sgn } \lambda} \right)
\]

where \( \Phi(\cdot) \) is the CDF of a standard normal.
The quantile, which is the inverse of the distribution function, is therefore

\[ F^{-1}_{-L}(q) = \lambda p \left( e^{\mu + sgn \lambda \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) \]

So can proceed to evaluate the index of satisfaction.

\[ -r_\varrho(\lambda) = \frac{1}{1 - c} \int_0^{1-c} \lambda p \left( e^{\mu + sgn \lambda \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq \]

\[ = \lambda p \left( \frac{1}{1 - c} \int_0^{1-c} e^{\mu + sgn \lambda \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right) \]

\[ = \lambda p \left( \frac{1}{1 - c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + sgn \lambda \sqrt{\Sigma} \phi(z)} dz - 1 \right) \]

where the last line is achieved by the change of variable \( z = \Phi^{-1}(q) \) and \( \phi(z) = \Phi'(z) \) is the density of a standard normal.

Since

\[ e^{\mu + sgn \lambda \sqrt{\Sigma} \phi(z)} = e^{\mu + \frac{1}{2} \Sigma \phi(z - sgn \lambda \sqrt{\Sigma})} \]

we have the final result,

\[ -r_\varrho(\lambda) = \lambda p \left( e^{\mu + \frac{1}{2} \Sigma} \frac{1}{1 - c} \Phi \left( \Phi^{-1}(1-c) - sgn \lambda \sqrt{\Sigma} \right) - 1 \right) \quad (1) \]

### 2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

\[ -r_\varrho(\lambda) \approx \lambda p \left( \mu - sgn \lambda \frac{\Phi^{-1}(1-c)}{1 - c} \sqrt{\Sigma} \right) \quad (2) \]

Let us spend a moment interpreting this. A long is less risky (\( \lambda > 0 \)) if the asset has a positive expected return (\( \mu > 0 \)), and a short (\( \lambda < 0 \)) is less risky if the asset has a negative expected return (\( \mu < 0 \)). In contrast, positive variance increases risk for any non-zero position.

This all seems quite reasonable for a rational risk measure.

### 3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the loss. In a Delta-Gamma setting, we can replace the objective by the quadratic

\[ -L = \lambda p \left( e^X - 1 \right) \approx \lambda p \left( X + \frac{1}{2} X^2 \right) \]

hence \( \Theta_\lambda = 0 \), \( \Delta_\lambda = \lambda p \), and \( \Gamma_\lambda = \lambda p \). Let us define a new objective to represent this approximation.

\[ \Xi_\lambda = \lambda p \left( X + \frac{1}{2} X^2 \right) \]
Is is straightforward to work out that the first several central moments of this are

\[
E(\Xi_\lambda) = \lambda p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right)
\]

\[
Sd(\Xi_\lambda) = |\lambda| p \sqrt{\Sigma} \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma}
\]

\[
Sk(\Xi_\lambda) = 3 \text{sgn} \lambda \sqrt{\Sigma} \frac{(1 + \mu)^2 + \frac{1}{3} \Sigma}{((1 + \mu)^2 + \frac{1}{2} \Sigma)^{3/2}}
\]

The third-order Cornish-Fisher expansion for expected shortfall in general is

\[
-r_\varphi(\lambda) \approx E(\Xi_\lambda) + Sd(\Xi_\lambda) \left( z_1 + \frac{z_2 - 1}{6} Sk(\Xi_\lambda) \right)
\]

with coefficients

\[
z_1 = \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q) \, dq = -\frac{\phi \left( \Phi^{-1}(1 - c) \right)}{1 - c}
\]

\[
z_2 = \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q)^2 \, dq = 1 - \frac{\phi \left( \Phi^{-1}(1 - c) \right)}{1 - c} \Phi^{-1}(1 - c)
\]

depending on the confidence level \( c < 1 \).

Putting this together, we get a third expression for the index of satisfaction.

\[
-r_\varphi(\lambda) \approx \lambda p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) - |\lambda| p \frac{\phi \left( \Phi^{-1}(1 - c) \right)}{1 - c} \sqrt{\Sigma}
\]

\[
\cdot \left( \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \text{sgn} \lambda \frac{(1 + \mu)^2 + \frac{1}{3} \Sigma \Phi^{-1}(1 - c) \sqrt{\Sigma}}{(1 + \mu)^2 + \frac{1}{2} \Sigma} \right)
\]

This result agrees with (2) to lowest order in \( \mu \) and \( \Sigma \).

\(^1\)The trick to these integrals is to realize that \( \phi'(z) = -z \phi(z) \).
4 Modeling Default

Our horizon asset value $P$ is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$-L' = \lambda p (Ye^X - 1)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before$^2$, but now we add an independent default indicator $Y \sim \text{Bern}(1 - h)$ for default probability $h$.

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$^2$Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.