Quantitative Risk Management
Case for Week 4 (Spring)

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Let us consider the expected shortfall index of satisfaction for a very simple portfolio: λ shares in an asset whose value today is $p > 0$ and whose horizon value P is lognormal.

Let us assume that the objective measure is mark-to-market profit; therefore in the text's notation, we have (apologies for the signs)

$$
\begin{aligned}
-L &= \lambda \left(P - p \right) \\
&= \lambda p \left(e^X - 1 \right)
\end{aligned}
$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean μ and variance $\Sigma > 0$. The risk measure is

$$
-\varrho(L) = -r_\varrho(\lambda) = \frac{1}{1-c} \int_0^{1-c} F_{-L}^{\leftarrow}(q) \, dq
$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$
\begin{aligned}
F_{-L}(z) &= \mathrm{P} \left\{ -L < z \right\} \\
&= \mathrm{P} \left\{ \lambda p \left(e^X - 1 \right) < z \right\} \\
&= \mathrm{P} \left\{ X \operatorname{sgn} \lambda < \log \left(1 + \frac{z}{\lambda p} \right) \operatorname{sgn} \lambda \right\} \\
&= \mathrm{P} \left\{ \frac{X - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \lambda < \frac{\log \left(1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma}} \operatorname{sgn} \lambda \right\} \\
&= \Phi \left(\frac{\log \left(1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma} \operatorname{sgn} \lambda} \right)
\end{aligned}
$$

where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the distribution function, is therefore

$$F_{-L}^{\leftarrow}(q) = \lambda p \left(e^{\mu + \text{sgn}\,\lambda \sqrt{\Sigma}\Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$
\begin{aligned}
-r_\varrho(\lambda) &= \frac{1}{1-c} \int_0^{1-c} \lambda p \left(e^{\mu + \text{sgn}\,\lambda \sqrt{\Sigma}\Phi^{-1}(q)} - 1 \right) \, dq \\
&= \lambda p \left(\frac{1}{1-c} \int_0^{1-c} e^{\mu + \text{sgn}\,\lambda \sqrt{\Sigma}\Phi^{-1}(q)} \, dq - 1 \right) \\
&= \lambda p \left(\frac{1}{1-c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn}\,\lambda \sqrt{\Sigma}z} \phi(z) \, dz - 1 \right)
\end{aligned}
$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \text{sgn}\,\lambda \sqrt{\Sigma}z} \phi(z) = e^{\mu + \frac{1}{2}\Sigma} \phi \left(z - \text{sgn}\,\lambda \sqrt{\Sigma} \right)$$

we have the final result,

$$-r_\varrho(\lambda) = \lambda p \left(e^{\mu + \frac{1}{2}\Sigma} \frac{1}{1-c} \Phi \left(\Phi^{-1}(1-c) - \text{sgn}\,\lambda \sqrt{\Sigma} \right) - 1 \right) \tag{1}$$

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$-r_\varrho(\lambda) \approx \lambda p \left(\mu - \text{sgn}\,\lambda \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \sqrt{\Sigma} \right) \tag{2}$$

Let us spend a moment interpreting this. A long is less risky ($\lambda > 0$) if the asset has a positive expected return ($\mu > 0$), and a short ($\lambda < 0$) is less risky if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance increases risk for any non-zero position.

This all seems quite reasonable for a rational risk measure.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the loss. In a Delta-Gamma setting, we can replace the objective by the quadratic

$$-L = \lambda p \left(e^X - 1 \right) \approx \lambda p \left(X + \tfrac{1}{2} X^2 \right)$$

hence $\Theta_\lambda = 0$, $\Delta_\lambda = \lambda p$, and $\Gamma_\lambda = \lambda p$. Let us define a new objective to represent this approximation.

$$\Xi_\lambda = \lambda p \left(X + \tfrac{1}{2} X^2 \right)$$
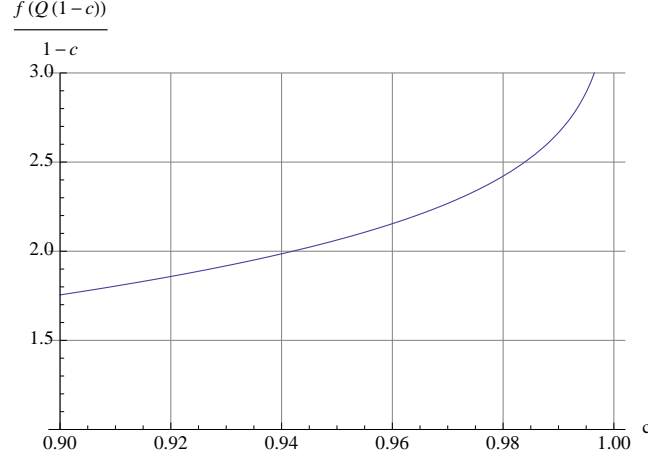
Is is straight-forward to work out that the first several central moments of this are

$$\mathrm{E}\left(\Xi_\lambda\right) = \lambda p \left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma\right)$$

$$\mathrm{Sd}\left(\Xi_\lambda\right) = |\lambda| p \sqrt{\Sigma}\sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma}$$

$$\mathrm{Sk}\left(\Xi_\lambda\right) = 3\,\mathrm{sgn}\,\lambda\sqrt{\Sigma}\,\frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{\left((1+\mu)^2 + \tfrac{1}{2}\Sigma\right)^{3/2}}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$-r_\varrho(\lambda) \approx \mathrm{E}\left(\Xi_\lambda\right) + \mathrm{Sd}\left(\Xi_\lambda\right)\left(z_1 + \frac{z_2 - 1}{6}\,\mathrm{Sk}\left(\Xi_\lambda\right)\right)$$

with coefficients

$$z_1 = \frac{1}{1-c}\int_0^{1-c}\Phi^{-1}(q)\,dq = -\frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c}$$

$$z_2 = \frac{1}{1-c}\int_0^{1-c}\Phi^{-1}(q)^2\,dq = 1 - \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c}\Phi^{-1}(1-c)$$

depending on the confidence level $c < 1$[1].

Putting this together, we get a third expression for the index of satisfaction.

$$-r_\varrho(\lambda) \approx \lambda p\left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma\right) - |\lambda| p\frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c}\sqrt{\Sigma}$$
$$\cdot\left(\sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma} + \tfrac{1}{2}\,\mathrm{sgn}\,\lambda\frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{(1+\mu)^2 + \tfrac{1}{2}\Sigma}\Phi^{-1}(1-c)\sqrt{\Sigma}\right) \quad (3)$$

This result agrees with (2) to lowest order in μ and Σ.

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[1] The trick to these integrals is to realize that $\phi'(z) = -z\phi(z)$.

4 Modeling Default

Our horizon asset value P is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$-L' = \lambda p \left(Y e^X - 1 \right)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before[2], but now we add an independent default indicator $Y \sim \mathrm{Bern}(1 - h)$ for default probability h.

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[2]Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.