Probability Distributions

MFM Practitioner Module:
Quantitative Risk Management

John Dodson

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Definitions

For our purposes, a random variable is a quantity whose value is not known to us right now (but may be at some point in the future). These can be represented mathematically as measurable functions of a sample space, and we will usually denoted them by upper-case letters. We will usually denote a particular value obtained by a random variable by the corresponding lower-case letter.

There must be a probability associated with every set of outcomes. Such sets are called events and might consist of intervals or points or a combination thereof. The corresponding probability is called the probability measure of the event.

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This structure lends itself to a measure theory interpretation, where the probability associated with a set is simply the integral of the probability density over that set. For reals $a < b$,

$$\mathrm{P}\left\{X : a < X < b\right\} = \int_a^b f_X(x)\, dx$$

If the sample space of X is the real numbers, $\mathbb{R}$, then $f_X(\cdot)$ must have certain properties: It must be a non-negative (generalized) function and

$$\lim_{x\to-\infty} f_X(x) = 0 \qquad \lim_{x\to+\infty} f_X(x) = 0$$

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$$

Characterizations

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In addition to the density function, there are several other equivalent characterizations of a real random variable

- distribution function $F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$
- quantile function $q_p(F_X) = F_X^{\leftarrow}(p)$
- characteristic function $\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx$

The quantile function is defined in terms of the generalized inverse

$$F^{\leftarrow}(p) = \inf\{x : F(x) \geq p\}$$

The characteristic function is based on the Fourier transform of the density, where $i^2 \triangleq -1$.

Functions of random variables are generally other random variables. Evaluating them amounts to determining the effect on a characterization.

Taxonomy of Random Variables

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Below is a taxonomy of common random variables, classified primarily by the topology of their support.

- finite
  - Dirac
  - Bernoulli
- countable
  - binomial
  - geometric
  - Poisson
- interval
  - uniform
  - beta
- half-line
  - exponential
  - Gamma
- unbounded
  - normal
  - Cauchy
  - (Lévy's) stable
- *transforms*
  - generalized Pareto
  - gen. inv. Gaussian
  - lognormal
- *mixtures*
  - (Gosset's) Student's-t
  - negative-binomial
  - generalized hyperbolic
- *non-parametric*
  - empirical

Finite Support

Let us start our tour by considering two special classes of random variables.

Bernoulli

A Bernoulli r.v. is a "bit". Its sample space can be characterized by

- 0/1, T/F, heads/tails, win/lose, up/down

and there are only four possible events (what are they?). It has only one parameter (not necessarily equal to $\frac{1}{2}$).

Dirac

The sample space for a Dirac r.v. is, in principle, $\mathbb{R}$; but only events that contain x_0 have non-zero measure, where x_0 is the only parameter. It can be thought of as the degenerate continuous r.v. We write its density as $f_X(x) = \delta(x - x_0)$, which can be represented as a spike.

Countable Support: Discrete Sample Space

binomial

If we think of a Bernoulli r.v. as having sample space $\{0, 1\}$, the sum of n Bern(p) r.v.'s is a bin(n, p) and its sample space is $\{0, 1, \ldots, n\}$. From combinatorics, we know

$$\mathrm{P}\{i\} = \binom{n}{i} p^i (1-p)^{n-i} \quad \forall i \in \{0, 1, \ldots, n\} \subset \mathbb{Z}$$

Stirling's Approximation

A useful result from calculus is Stirling's approximation, which says that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for $n \gg 1$. In particular,

$$\binom{n}{i} \approx \frac{1}{\sqrt{2\pi n}} \left(\frac{i}{n}\right)^{-i-\frac{1}{2}} \left(1 - \frac{i}{n}\right)^{-n+i-\frac{1}{2}}$$

for large n, i.

Countable Support: Discrete Sample Space

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geometric

A geometric r.v. is also related to a sequence of Bernoulli r.v.'s. In terms of the coin toss analogy, it is the length of the "streak" of tails tossed before the next head appears. Again from combinatorics, we know

$$\mathrm{P}\{i\} = p(1-p)^i \quad \forall i \in \{0, 1, \ldots\}$$

- Notice that the sample space here is countably infinite. One could observe a streak of any length for $0 < p < 1$.
- Notice also that the process underlying this model is memoryless. The fact that one has already observed a streak of length n is irrelevant: the only parameter is p, the chance of breaking the streak on the next toss.

Countable Support: Discrete Sample Space

Another notable r.v. whose sample space is the nonnegative integers is the Poisson.

Poisson

Consider a binomial r.v. with a very large sample space but a very small probability of occurrence. If we take the limit $n \to \infty$ but we fix $p = \lambda/n$, we get

$$\mathrm{P}\{i\} = e^{-\lambda} \frac{\lambda^i}{i!}$$

using Stirling's Approximation and the limit definition of the exponential function,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x \quad \forall x \in \mathbb{R}$$

Analogously, we shall see later that the interval *between* rare events is the limiting case of a geometric r.v.

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Now let's move on the r.v.'s whose sample space is a segment of the real line, traditionally taken to be $[0, 1]$.

uniform

The probability of observing a value of a uniform r.v. between $0 \leq a < b \leq 1$ is equal to $b - a$.

- Any particular value between zero and one is equally likely to be observed.
- imagine a binary decimal, e.g. $0.0110\ldots$, where each bit to the right of the decimal place is Bern $\left(\frac{1}{2}\right)$. This is a uniform r.v.
- All modern computer systems can generate an almost-endless stream of uniform (pseudo)-random variates, which can be used for generating samples of other r.v.'s using transformation techniques.

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beta

A beta r.v. is parameterized by two continuous parameters, $\alpha, \beta > 0$. The scale factor for the density involves the Gamma function.

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \forall x \in [0,1]$$

There is a deep connection between the beta and binomial. The formulæ for the densities are essentially the same; but instead of describing the count, the beta describes the probability.

- The beta is a good model for an unknown probability.
- The beta is also a good model for an unknown fraction.
- The uniform is a special case, beta$(1,1) \sim$ U$([0,1])$.

Half-line Support

exponential

The limiting case of the geometric with $p = \lambda/n$ and $n \to \infty$ is the exponential. Its distribution function is simply

$$F_X(x) = 1 - e^{-\lambda x} \quad \forall x \geq 0$$

Differentiating, we get the density.

$$f_X(x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$$

- Note that the mode of an exponential r.v. is zero.
- The minimum of n $\exp(\lambda)$ r.v.'s is $\exp(n\lambda)$
- The interval between arrivals of a Poisson process is exponential.

Half-line Support

One can arrive at a Gamma r.v. by several routes. We will approach it as a sum of exponentials.

Gamma

Since the characteristic function of $Y \sim \exp(\lambda)$ is $\phi_Y(t) = \left(1 - \frac{it}{\lambda}\right)^{-1}$ (prove), the characteristic function of the sum of k exponential r.v.'s is $\phi_{Y_1+\ldots+Y_k}(t) = \left(1 - \frac{it}{\lambda}\right)^{-k}$. We can apply the Fourier transform to get the density of $X = Y_1 + \ldots + Y_k$,

$$f_X(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} \quad \forall x \geq 0$$

- This is also a sum of squared normals (Chi-squared)
- and a natural description for the space of random positive-definite matrices (Wishart)

Unbounded Support

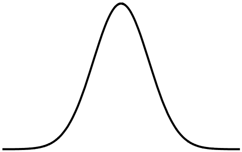
The most important distribution for $X \in \mathbb{R}$ is the normal or Gaussian distribution, $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ with μ the mean and σ^2 the variance (σ is the standard deviation).

$$f_X(x) = \tfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F_X(x) \triangleq \Phi(\frac{x-\mu}{\sigma}) = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

$$q_p\left(F_X\right) = \mu + \sigma\,\Phi^{-1}(p) = \mu + \sigma\sqrt{2}\,\mathrm{erf}^{-1}\left(2p-1\right)$$

$$\phi_X(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$



Normal density

Unbounded Support

Not every r.v. has a finite variance. The simplest example of an unbounded r.v. without a finite variance is the Cauchy.

Cauchy

The standard version[1] of the Cauchy has the density

$$f_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$$

whose graph looks like that of a normal; but statistically it is nothing like a normal.

- The standard Cauchy has the characteristic function $\phi_X(t) = e^{-|t|}$.
- The ratio of two normal r.v.'s is a Cauchy.

[1] Any affine transformation of a Cauchy is still a Cauchy.

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(Lévy's) stable

The Cauchy is a special case of the (Lévy's) stable family. Not only do stable r.v.'s lack a variance, but (excepting the Cauchy) they also lack a tractable density.
The standard stable characteristic function is

$$\phi_X(t) = \begin{cases} e^{-|t|^\alpha(1-\operatorname{sgn}(t) i\beta \tan(\alpha\pi/2))} & \alpha \neq 1 \\ e^{-|t|(1+\operatorname{sgn}(t) i\beta (2/\pi)\log|t|)} & \alpha = 1 \end{cases}$$

for parameters $0 < \alpha < 2$ and $-1 < \beta < 1$.

- The Cauchy corresponds to $\alpha = 1$ and $\beta = 0$
- $\beta \gtrless 0$ allows for the density to by asymmetric.
- The limit $\alpha \to 2$ is $\mathcal{N}(0, 2)$. The limit $\alpha \to 0$ is delta(0).

Common Transforms

Pareto

A basis for the concept of power laws is a result about the frequency of extreme events. This has at its heart a (generalized) Pareto r.v. which is an exponential of an exponential, $\log(1+\xi X) \sim \exp(1/\xi)$.
The resulting distribution function is

$$F_X(x) = 1 - (1+\xi x)^{-1/\xi} \quad \forall \begin{cases} 0 \le x & \xi \ge 0 \\ 0 \le x \le -\frac{1}{\xi} & \xi < 0 \end{cases}$$

- ξ is sometimes called the tail index
- for $\xi = 0$, the limit gives $F_X(x) = 1 - e^{-x}$
- note that if $U \sim \mathrm{U}([0,1])$, then $\frac{U^{-\xi}-1}{\xi} \sim \mathrm{Pareto}(\xi)$

Common Transforms

The gamma is a reasonable model for an unknown scale factor. But sometimes your application will call for dividing rather than multiplying.

Reciprocal Gamma

It is a simple matter to evaluate the density of the reciprocal of a gamma r.v.,

$$f_X(x) = \frac{\lambda^k}{\Gamma(k)} x^{-1-k} e^{-\lambda/x} \quad \forall x > 0$$

for parameters $\lambda > 0$ and $k > 0$.

- the reciprocal gamma comes up in Bayesian analysis
- it is a natural description for an unknown variance

Common Transforms

Since most assets cannot become liabilities (their values cannot be negative), the support for future asset values (or prices) is $\mathbb{R}^+$ (or some subset). Furthermore, since holdings values are usually price $\times$ quantity, nominal per-share prices are rarely important. Hence a natural model for the future value of an asset is a lognormal r.v.

lognormal

The logarithm of a lognormal r.v. is normal. It has a density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{e^{-\frac{\log(x/\mu)^2}{2\sigma^2}}}{x} \quad \forall x > 0$$

for parameters $\mu > 0$ and $\sigma > 0$.

- Note that, unlike the normal, the expected value of a lognormal r.v. involves both parameters: $\mathrm{E}\,X = \mu e^{\frac{1}{2}\sigma^2}$.

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The gamma and reciprocal gamma can be combined into a generalized family for r.v.'s with half-line support.

Generalized Inverse Gaussian

$$f_X(x) = \frac{\chi^{-\lambda}\left(\sqrt{\chi\psi}\right)^{\lambda}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} x^{\lambda-1} e^{-\frac{\chi}{2x}-\frac{\psi x}{2}}$$

for $x > 0$, where $K_\cdot(\cdot)$ is Bessel's function of the third kind.

- There are several versions of parameterization in use
- Other members of this family include the inverse Gaussian and the reciprocal inverse Gaussian
- The name relates to the first passage time of a Brownian motion through a boundary

Mixtures

Conditioning generally reduces entropy. Mixing has the opposite effect. This is useful if you want to moderate overconfidence or statistical hubris.

Say X is the parameteric r.v. we are interested in.

1. Concatenate it with some function of the parameters to make a multivariate r.v. $X+\!\!\!+\Theta$.
2. Specify the marginal density of Θ and the conditional density of $X|\Theta$.
3. Integrate over the support of Θ to get the marginal density of X, the new mixture.

$$f_{X+\!\!\!+\Theta}(x+\!\!\!+\theta) = f_{X|\Theta}(x;\theta) f_\Theta(\theta)$$
$$f_X(x) = \int_{\Omega(\Theta)} f_{X|\Theta}(x;\theta) f_\Theta(\theta)\, d\theta$$

Common Mixtures

If you are working with an r.v. that is Poisson, but you only have an estimate for the parameter, one approach is to say that the true parameter is a draw of a gamma r.v. which you can confidently characterize.

negative binomial

This is called the gamma-Poisson mixture, and the result is called the negative binomial. The hierarchical model is

$$X|\lambda \sim \text{Poisson}(\lambda)$$
$$\lambda \sim \text{Gamma}\left(k, \frac{p}{1-p}\right)$$

with the result

$$P_X\{i\} = \frac{\Gamma(i+k)}{i!\Gamma(k)} p^k (1-p)^i \quad \forall i \in \{0, 1, \ldots\}$$

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Student's-t is a symmetric r.v. which exhibits leptokurtosis.

(Gosset's) Student's-t

Consider a normal r.v. with an unknown variance close to one. If the variance is a draw from an reciprocal gamma,

$$X \left| \sigma^2 \right. \sim \mathcal{N}\left(0, \sigma^2\right)$$
$$\sigma^2 \sim \text{Gamma}^{-1}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$$

the resulting unconditioned density is

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{1}{\sqrt{\nu}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

- $\nu = 1$ is a Cauchy
- The limit $\nu \to \infty$ is a normal

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The version of the Student's-t above has a variance for $\nu > 2$, but it is not unity.

Standardized Student's-t

The standardized version is useful for fitting residuals. It has the density

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{1}{\sqrt{\nu - 2}} \left(1 + \frac{x^2}{\nu - 2}\right)^{-\frac{\nu+1}{2}}$$

- Note that $\mathrm{E}\, e^X \to \infty$ for any finite ν. *This would seem to be a problem for a model of log-returns!*
- For historical reasons, if the parameter $\nu > 0$ is an integer, it is termed the degrees of freedom.

Common Mixtures

The Student's-t is part of the generalized hyperbolic family, which is based on a normal mean–generalized inverse Gaussian variance mixture (with $\lambda = -\nu/2$).

Normal-reciprocal inverse Gaussian (NRIG)

The symmetric generalized hyperbolic r.v. with $\lambda = \frac{1}{2}$, is particularly useful. The standardized version has the density

$$f_X(x) = \frac{1}{\pi} e^g \sqrt{1+g}\, K_0\left(\sqrt{g^2 + (1+g)x^2}\right)$$

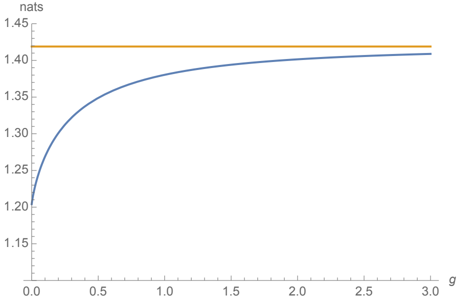
in the Babbs representation for shape parameter $g \geq 0$. It is not obvious, but the limit $g \to \infty$ corresponds to the normal.

- As a model for the residuals of the log-returns for asset prices, this is superior to the Student's-t example from the text because $\mathrm{E}\, e^X$ is finite(!)

Common Mixtures

We can use the NRIG to illustrate the effects of scaling and mixing on entropy.

$$H_X = \mathrm{E}\, H_{X|\Theta}(\Theta) + H_\Theta = H_{aX} - \log|a|$$



Entropy for a standard NRIG & a standard Normal

- A normal with the same entropy as an NRIG would have a standard deviation up to about 24% smaller.

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A modern trend in statistics is to move away from parametric descriptions towards non-parametric descriptions.

empirical

The most natural non-parametric description of a r.v. X based on a dataset $\{x_1, x_2, \ldots, x_n\}$ is the empirical r.v.

$$f_X(x) = \frac{1}{n} \sum_{i=1}^{n} \delta\left(x - x_i\right)$$

$$F_X(x) = \frac{1}{n} \sum_{i=1}^{n} H\left(x - x_i\right)$$

- This can be regularized by replacing the Dirac deltas by normal densities with sufficiently small variances, which is termed kernel smoothing.

Multivariate Distributions

Many of the univariate distributions can be generalized to a vector-valued sample space

- stable family and related, including the normal, Cauchy, Student's-t, and generalized hyperbolic
- discrete and empirical
- uniform

For the uniform, this is just a matter of defining the support in $\mathbb{R}^n$ and normalizing appropriately. For the others, often the replacement

$$\left(\frac{X-\mu}{\sigma}\right)^2 \to (X-\mu)'\Sigma^{-1}(X-\mu)$$

and normalization is all that is required to lift the sample space from $\mathbb{R}$ to $\mathbb{R}^N$.

- A multivariate density can always be used as a basis for a copula.

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