

Extreme Value Theory

MFM Practitioner Module: Quantitative Risk Management

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The n -block maxima¹ is a random variable defined as

$$M_n \triangleq \max(X_1, \dots, X_n)$$

for i.i.d. random variables X_i with distribution function $F(\cdot)$. We are interested in $n \rightarrow \infty$. If there exists a sequence of **normalizations** of M_n (with $c_n > 0$) such that

$$\lim_{n \rightarrow \infty} F^n(c_n x + d_n)$$

converges to a non-degenerate distribution function, $H(x)$, then

$$H(x) = \begin{cases} \exp(-(1 + \xi x)^{-1/\xi}) & \xi > 0 \text{ and } x \geq -1/\xi \\ \exp(-e^{-x}) & \xi = 0 \\ \exp(-(1 + \xi x)^{-1/\xi}) & \xi < 0 \text{ and } x < -1/\xi \end{cases}$$

¹Use $1 - F(-x)$ and $-\max(-X_1, \dots, -X_n)$ for the minima.

Maxima

This remarkable result, the Fisher–Tippett–Gnedenko theorem (1927–28/1943), is analogous to the **central limit theorem** for an appropriately normalized $S_n \triangleq \sum_{i=1}^n X_i$:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}} S_n - \sqrt{n}\mu \right) \sim \mathcal{N}(0, \sigma^2)$$

Generalized Extreme Value Distribution

$H(\cdot)$ from above is called the **generalized extreme value distribution** and it has a single parameter ξ .

Types

The GEV is continuous in ξ for each x , but its sign has can be used as a classifier

- ▶ $\xi > 0$ is the **Fréchet** with finite moments to order $1/\xi$.
- ▶ $\xi = 0$ is the **Gumbel** with finite moments of all orders.
- ▶ $\xi < 0$ is the **Weibull** with a finite **right endpoint**.

The GEV result is about i.i.d. sequences of r.v.'s. We have seen that timeseries of innovations of financial invariants are independent, but not generally identically-distributed.

Stationary Time Series

If the normalized n -block maxima of the **associated strict white noise** for a stationary time series process has a limiting distribution $H(\cdot)$ in one of the GEV classes, then there exists $0 < \theta \leq 1$ such that the limit of the normalized n -block maxima of the innovations is $H^\theta(\cdot)$.

- ▶ In particular, the n -block maxima of the innovations can be re-normalized to yield the same GEV distribution as the associated white noise with the same ξ .
- ▶ Effectively $\tilde{n} = \theta n$ in the limit, which can be thought of as representing **clustering** in the extreme values of the innovations for $\theta < 1$.

Maxima

Threshold
Exceedances

Estimators

EVT Loss
Distribution

Threshold Exceedances

Typically we are not as interested in the n -block maxima as we are in the relative frequency of a range of extreme outcomes. A result in this regard leads to the **generalized Pareto** distribution that we have already seen.

Generalized Pareto

The distribution function with scale parameter $\beta > 0$ is

$$G(x) = \begin{cases} 1 - (1 + \xi x/\beta)^{-1/\xi} & \xi > 0 \text{ and } x \geq 0 \\ 1 - e^{-x/\beta} & \xi = 0 \text{ and } x \geq 0 \\ 1 - (1 + \xi x/\beta)^{-1/\xi} & \xi < 0 \text{ and } 0 \leq x < -\beta/\xi \end{cases}$$

Excess Distribution

If r.v. X has distribution $F(\cdot)$, the **excess distribution** is

$$F_{\eta}(x) \triangleq \text{P}[X - \eta \leq x | X > \eta] = \frac{F(x + \eta) - F(\eta)}{1 - F(\eta)}$$

Pickands–Balkema–de Haan theorem (1974–75)

Iff X is GEV with parameter ξ and right endpoint x_F , then there exists $\beta(\eta) > 0$ such that

$$\lim_{\eta \rightarrow x_F} \sup_{0 \leq x < x_F - \eta} |F_\eta(x) - G(x; \xi, \beta(\eta))| = 0$$

That is, as the threshold level is raised, the excess distribution becomes arbitrarily close to a generalized Pareto distribution with the same shape parameter as the GEV.

Mean Excess

Note that in the limit $\eta \rightarrow x_F$, $\beta(\eta)$ becomes linear. Since $E[X - \eta | X > \eta] = \beta(\eta)/(1 - \xi)$, the **mean excess** also becomes linear in the threshold for $\xi < 1$.

Generalized Extreme Value

It is possible to estimate the shape of the limiting GEV from a sample of n -block maxima of i.i.d. data using maximum likelihood, but this is very inefficient, since you effectively reduce all but a fraction $1/n$ of your data to partial ranks.

Generalized Pareto

In the GP setting you can use a higher fraction of the data depending on how you choose the threshold; and we can make use of results such as the limiting linearity of the mean excess above to help set it.

Exceedance Point Process

In a strict white noise setting, the interval between threshold exceedances is an exponential r.v. in the threshold limit, so we can use even more of the data taking that into account.

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We have already worked with the MLE for GP. In this setting, we form the log-likelihood of the GP approximation of the threshold excess

$$\begin{aligned} \log L(\xi, \beta) = & \\ & - N_\eta \log \beta - \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^{N_\eta} \log \left(1 + \xi \frac{x_{(N-N_\eta+i)} - \eta}{\beta}\right) \end{aligned}$$

where N_η is the number of observations that exceed η and the N observations are indexed by their ranks. The estimator is of course

$$\left(\hat{\xi}, \hat{\beta}\right) = \arg \max_{(\xi, \beta)} \log L(\xi, \beta)$$

Hill Estimator

In the Fréchet class, for $\xi > 0$, the distribution function has a tail of the form $\bar{F}(x) = x^{-1/\xi} L(x)$ for some **slowly varying** function $L(\cdot)$.

Hill Estimator

The Hill estimator is based on the observation about the mean excess of the logarithm of X .

$$E[\log X - \log \eta | X > \eta] \approx \frac{L(\eta)\eta^{-1/\xi}\xi}{\bar{F}(\eta)} \approx \xi$$

in the limit $\eta \rightarrow \infty$ from Karamata's theorem. The estimator is hence

$$\hat{\xi}_k^{(H)} = \frac{1}{k} \sum_{i=1}^k \log x_{(n-k+i)} - \log x_{(n-k)}$$

in terms of the observations indexed by their ranks.

Points over Thresholds

In this model, the order of the data matter. Let $t = i/n$ for $i = 1, \dots, n$. Define a **state space** \mathcal{X} for (t, x) . The **marked point process** defined by X exceeding some high threshold η before t has intensity rate

$$\tau(x) \triangleq H_{\xi, \mu, \sigma}(x)$$

where σ, μ represent c_n, d_n , and excess magnitude

$$\bar{F}_\eta(x) = \bar{G}_{\xi, \beta}(x)$$

for scale $\beta = \sigma + \xi(\eta - \mu)$.

Estimator

The likelihood function for this model is given by

$$\log L(\xi, \sigma, \mu) = \log L_{GP}(\xi, \sigma; x - \eta) - \tau + N_\eta \log \tau$$

hence $\hat{\tau} = N_\eta$; and from the MLE of the GP, we can infer $\hat{\mu}, \hat{\sigma}$.

The convergence of excess loss to a GP random variable can be used to calculate VaR and ES. In particular, we can set the (right) tail mass θ to some sufficiently small value, then

$$F_L(x) \approx \begin{cases} ? & x < \eta \\ 1 - \theta \left(1 + \xi \frac{x-\eta}{\beta}\right)^{-1/\xi} & (\xi > 0 \text{ and } x \geq \eta) \text{ or} \\ & (\xi < 0 \text{ and } \eta \leq x < \eta - \frac{\beta}{\xi}) \\ 1 & \xi < 0 \text{ and } x \geq \eta - \frac{\beta}{\xi} \end{cases}$$

and we can invert this to get

$$\text{VaR}_\alpha \approx \eta + \frac{\beta}{\xi} \left(\left(\frac{\theta}{1-\alpha} \right)^\xi - 1 \right) \quad \text{for } 1 - \theta \leq \alpha < 1$$

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Furthermore, we can integrate the quantile function to get expected shortfall (for $\xi < 1$),

$$ES_{\alpha} \approx \eta + \frac{\beta}{\xi} \left(\frac{1}{1-\xi} \left(\frac{\theta}{1-\alpha} \right)^{\xi} - 1 \right) \quad \text{for } 1 - \theta \leq \alpha < 1$$

Recalling our previous comparison of VaR and ES in terms of the Cornish-Fisher moment expansion, we again see that there is a fundamental equivalence to the extent that $\xi \approx 0$.

$$\lim_{\alpha \rightarrow 1} \frac{ES_{\alpha}}{VaR_{\alpha}} = \frac{1}{1-\xi}$$