Quantitative Risk Management
Spring Assignment

John Dodson

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This assignment is not a regular homework. It is worth half of the module grade for the spring term. Please share your solution with me through Google Drive before the beginning of the classroom session on Wednesday, March 1.

Please write up your results into a report and submit a PDF version. (My edition of LibreOffice can recognize most word processing formats; but sometimes the formatting, equations, and exhibits are altered or corrupted.) You may be able to covert directly to PDF from your word processor or type setter, or you may be able to "print to PDF" using a print driver.

I expect you to work independently and to **cite your sources** even if those sources include your classmates. Please remind yourself of the University's definition of scholastic dishonesty (see the syllabus).

Stress-Test Risk Measures

For a homogeneous loss in risk factors which are the components of an elliptical random vector $\boldsymbol{X} \sim \mathrm{E}_d(\boldsymbol{\mu}, \Sigma, \psi(\cdot))$, a coherent risk metric

$$\varrho(L) = \varrho\left(m + \boldsymbol{\lambda}'\boldsymbol{X}\right) = m + r_\varrho(\boldsymbol{\lambda})$$

in terms of the fixed loss m and risk factor allocation $\boldsymbol{\lambda}$, can be expressed as the maximum realized loss over a set of risk factor scenarios that is independent of allocation,

$$\varrho(L) = \sup\left\{m + \boldsymbol{\lambda}'\boldsymbol{x} : \boldsymbol{x} \in S_\varrho\right\}$$

which make up the ellipsoid in $\mathbb{R}^d$

$$S_\varrho = \left\{\boldsymbol{x} : (\boldsymbol{x}-\boldsymbol{\mu})'\,\Sigma^{-1}\,(\boldsymbol{x}-\boldsymbol{\mu}) = \varrho(Y)^2\right\}$$

where $Y \sim S_1(\psi(\cdot))$, is a (univariate) random variable with the same characteristic generator as $\boldsymbol{X}$; that is $\mathrm{E}\, e^{itY} = \psi\left(t^2\right)$.

Finite Scenario Set Approximation

In practice, we may be interested in approximating the risk factor scenario set by some finite subset. This can be especially useful when the risk factors are highly co-dependent and when transparency and computational efficiency risk evaluation is important[1].

[1] The CME SPAN methodology for collateral requirements on futures margin accounts is a variant of this.

Market Risk of a Treasury Portfolio

Let's consider the one-day $\alpha = 95\%$ confidence value-at-risk measure, $\varrho(L) = F_L^{\leftarrow}(\alpha)$ where X is a model for the one-day mark-to-market loss on Treasury bonds as described in the exercise from January 18, 2017.

The statistical model we fit in that exercise is equivalent to a multivariate Cauchy, $\boldsymbol{X} \sim \mathrm{E}_8\left(-\hat{\boldsymbol{\mu}}, \hat{\Sigma}, \psi(\cdot)\right)$ where

$$\psi\left(t^2\right) = e^{-|t|}$$

hence the distribution function of Y is $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y)$.

We are interested in approximating the value-at-risk with a finite risk factor stress-test set, where the (fixed) scenarios are drawn randomly[2] from the ellipsoid S_ϱ.

$$\varrho_n(L) = \max\left\{m + \boldsymbol{\lambda}'\boldsymbol{x} : \boldsymbol{x} \in S_{\varrho_n}\right\}$$

where

$$S_{\varrho_n} \subset S_\varrho$$
$$|S_{\varrho_n}| = n$$

Problems

1. Based on the calibration from the January 18 exercise, what is the one-day 95%-confidence value-at-risk for a long position in the constant-maturity ten-year Treasury bond for the first business day after January 13, 2017? **(30 points)**

2. Simulate various versions of the finite scenario approximation with $n = 20$, and estimate the bias and inefficiency of that risk measure relative to the result above for the ten-year Treasury. **(40 points)**

3. How large do you think n should be such that $\varrho_n(L)$ is a good approximation or $\varrho(L)$ for an arbitrary portfolio? **(10 points)**

Grading Rubric

Thirty out of one hundred points will be based on the following criteria:

- You follow the instructions. **(5 points)**
- You include adequate citations. **(5 points)**
- Your write-up is clear and professional. **(10 points)**

[2] In practice one might project the vertices of some fixed spherical tiling, but this still has arbitrary degrees of freedom.

Solution

The 95% quantile of a Cauchy random variable is

$$\varrho(Y) = \tan\left(\pi\left(\frac{95}{100} - \frac{1}{2}\right)\right) \approx 6.31$$

and the mean and dispersion of the ten-year Treasury daily return from the earlier assignment is

$$\hat{\mu}_6 \approx 1.69 \times 10^{-4}$$
$$\hat{\Sigma}_{6,6} \approx 1.07 \times 10^{-5}$$

so the value-at-risk of \$1 in the ten-year Treasury bond is

$$\hat{\varrho}\left(X_6\right)/\$1 = -\hat{\mu}_6 + \varrho(Y)\sqrt{\hat{\Sigma}_{6,6}} \approx 0.0205$$

or **2.05%**, based on $\boldsymbol{\lambda} = \boldsymbol{e}_6 = (0,0,0,0,0,1,0,0)'$, the sixth standard basis vector, which corresponds to the ten-year CMT.

Note that we have to reverse the sign of the mean since we are modeling losses.

Also note that this result is probably too high, since the largest daily loss we have observed in the ten-year Treasury over the past four years (based on the dataset) is only 1.84%. This probably means that the Cauchy model has too much kurtosis.

To generate the points in the stress scenario set S_{ϱ_n}, start with points $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n\}$ on the unit sphere. We also need the square root of the dispersion, which we can take to be the lower-diagonal Cholesky matrix L such that $\Sigma = LL'$. Then

$$\boldsymbol{x}_i = \boldsymbol{\mu} + \varrho(Y)L\boldsymbol{z}_i$$

are valid stress scenarios. To see this, check the definition:

$$\begin{aligned}
&\left(\boldsymbol{x}_i - \boldsymbol{\mu}\right)'\Sigma^{-1}\left(\boldsymbol{x}_i - \boldsymbol{\mu}\right) \\
&= \left(\boldsymbol{z}_i'L'\varrho(Y)\right)\left(L'^{-1}L^{-1}\right)\left(\varrho(Y)L\boldsymbol{z}_i\right) \\
&= \varrho(Y)^2\boldsymbol{z}_i'\boldsymbol{z}_i \\
&= \varrho(Y)^2
\end{aligned}$$

since $\boldsymbol{z}_i'\boldsymbol{z}_i = 1$ for $i = 1, 2, \ldots, n$.

One approach to generating points on the unit sphere is to generate points from any convenient elliptical random variable, such as i.i.d. multivariate normal variates, and then rescale each by the reciprocal of its norm.

I coded the simulation in a MATLAB script.

```
%% exact version
k=tan(pi*(.95-1/2));
var=mu(6)+c*sqrt(Sigma(6,6));
%% one million finite versions
L=chol(Sigma);
n=20;
varn=nan(1000000,1);
```

```
for i=1:length(varn)
    z=randn(n,d);
    z=z./repmat(sqrt(sum(z.*z,2)),1,d); % unit vectors
    x=repmat(mu,n,1)+k*z*L;
    varn(i)=max(x(:,6));
end
bias=mean(varn)-var;
inef=std(varn);
```

and the result I got for the bias of the value-at-risk with $n = 20$ was **-0.0077** times the asset value, or -38% of the true value. The inefficiency was **0.0027**, or about one-third of the bias.

You should expect the bias to decrease for increased n, at least in the case where the points are distributed uniformly over the ellipse. But this seems to be very gradual, probably because the area of an 8-dimensional sphere is so high[3]. For $50\times$ more points, $n = 1000$, I still got a bias of -0.0023 or -11%. Perhaps interestingly the inefficiency, at 0.00075, is still about about one-third of the bias.

It is clear that $\varrho_n(L) \leq \varrho(L)$ for a fixed confidence level, and this is the source for the bias above. Can we translate that bias into the effective confidence level? From the analytic expression for the value-at-risk, we can infer that the mean result is equivalent to a value-at-risk at about 92.13%. Conversely, if we use a 96.84% confidence, the bias for $\boldsymbol{\lambda} = \boldsymbol{e}_6$ is close to zero.

If I increase the number of stress-test scenarios to one hundred, I can eliminate the 95% value-at-risk bias by using a 96.11% confidence. At one thousand scenarios I only need to use a 95.55% confidence.

But even in this case there is no guarantee that we will not be under-estimating the risk for any portfolio that does not correspond exactly to a stress-test scenario, and we are guaranteed to be over-estimating the risk (at a stated 95% confidence) for certain portfolios near to the the portfolios that correspond to the particular stress-test scenarios.

[3]The area of an eight-sphere is $\frac{1}{3}\pi^4 \approx 32.5$. The average radial distance between nearest neighbors of twenty uniform points is about 0.89 radians or 51°. The average radial distance between nearest neighbors of a thousand uniform points is about 0.47 radians or 27°.