Quantitative Risk Management
Case for Week 5 (Spring)

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February 14, 2018

Consider five-day 99%-confidence value-at-risk risk measure estimated using a Monte Carlo technique and a conditional calibration with a two-year look-back interval.

We want to design an experiment to back-test this estimator using a violation-based test. Say that the principal concern in designing this experiment is how many years of historical returns data to use. For example,

- How many years of historical index data is required to reject a hypothesis that value-at-risk exceptions are a Bernoulli process at the 5% significance level?
- How many years of historical index data is required to reject a hypothesis that value-at-risk exceptions are a Bernoulli process at the 1% significance level?

The challenge in testing the distribution of the statistic

$$I_{t+1} \triangleq \mathrm{I}_{\left\{L_{t+1} > VaR_t^{(\alpha)}\right\}}$$

for value-at-risk under the null hypothesis is that for α close to one (as it typically is) these statistics have a high probability of being zero. So you need a large sample. The question is, how large?

Certainly you would want $m > 100$ in the case of $\alpha = 0.99$, so that the expected value of M_m is at least one (since it can only be integers!). But at $m = 100$, there is still a $\alpha^m \approx 37\%$ change of legitimately getting zero exceptions from a particular back-testing sample.

Let me reprise the definition of test “significance”. This is the (frequentist) likelihood of rejecting a correct model. Statistical tests are defined in terms of pre-defined “domains for acceptance and rejection”, which are subsets of the sample space of the text statistic estimator. Here are are interested in the probability of events that would cause us to falsely reject the model, and how to limit them to under say 5% or 1%.

In the case of value-at-risk, the statistic we are interested in is the count of the exceptions. For m trials, the sample space is the integers from 0 to m. Some of these are in the acceptance range, some are not. The main issue is what to do about the zero. If it is in the rejection range, its probability contributes to the significance. You need to choose a trial size sufficiently large to make this probability small.

Assuming the independence property, the number of value-at-risk violations is a binomial random variable

$$M_m \triangleq \sum_{i=1}^{m} I_{t+i} \sim \mathsf{Bin}(m, 1-\alpha)$$

The Central Limit Theorem tells us that M_m can be approximated by a normal random variable for m large. But a problem with finite samples is that it may be highly skewed. In fact, M_m fails the Jarque-Bera test for normality, even in the limit $m \to \infty$, for $\alpha > 0.95$ (although this is probably an indictment of the Jarque-Bera test because we *know* that it is normal in the limit for any α!).

The probability mass function of M_m as a binomial random variable is

$$\mathrm{P}\left\{M_m = i\right\} = \frac{m!}{i!(m-i)!}(1-\alpha)^i \alpha^{m-i} \quad \forall i \in \{0, 1, \ldots, m\} \subset \mathbb{Z}$$

For m large, the factorials can be replaced by Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$\mathrm{P}\left\{M_m = i\right\} \approx \alpha^m \frac{\left[m\left(1/\alpha - 1\right)\right]^i}{i!} \frac{e^{-i}}{\left(1 - i/m\right)^{m-i+1/2}} \quad \forall i \in \{0, 1, \ldots, m\} \subset \mathbb{Z}$$

With two further approximations, we can conclude that M_m is approximated by a Poisson random variable: Firstly, note that the standard Maclaurin series for the logarithm implies that $1/\alpha - 1 \approx -\log\alpha$ for $\alpha \approx 1$. Secondly, since $m - i + 1/2 \approx m$, and $(1 - i/m)^m \approx e^{-i}$ for large m, the final quotient is approximately one.

So, with $\lambda \triangleq -m\log\alpha$,

$$\mathrm{P}\left\{M_m = i\right\} \approx e^{-\lambda}\frac{\lambda^i}{i!} \quad \forall i \in \mathbb{Z}$$

Ideally you would like the sample space to divide into three ranges: the acceptance region surrounding the median and equally-probable lower and upper rejection ranges. In this case, the lower range is the most challenging. The smallest possible lower rejection range is $\{M_m = 0\}$, and we know that the probability of that (false) outcome is α^m. So in the case of 5% significance we would want m big enough so that $\alpha^m < 2.5\%$. This means the minimum m at this significance is somewhere between 300 and 400.

At $m = 400$, the probability of getting a zero falls to under 2%; and there is about a 96% probability of getting a value of M_m that is between one and eight (the mean and the mode are both four). This is probably the weakest test we should consider if we want a 5% significance.

Since our analysis horizon is five business days, and we want to continue to rely on the independence assumption, we need to begin our back-test 2,000 business days, or about eight years, ago. Since the calculation itself requires a two-year look-back, we need about **ten years of historical data** in total for 5% significance.

For 1% significance, we need m large enough so that $\alpha^m < 0.5\%$. This means the minimum m is is somewhere between 500 and 600. For $m = 600$ the probability of getting M_m between one and thirteen is about 99.4%.

With non-overlapping five-day analysis horizon intervals, 600 trials requires 3,000 business days, or about twelve years. With the two-year look-back, that means we need about **fourteen years of historical data** for a 1% significance.

	$m = 100$		$m = 400$		$m = 600$	
i	binomial	Poisson	binomial	Poisson	binomial	Poisson
0	0.3660	0.3660	0.0180	0.0180	0.0024	0.0024
1	0.3697	0.3679	**0.0725**	**0.0722**	**0.0146**	**0.0145**
2	0.1849	0.1849	**0.1462**	**0.1451**	**0.0441**	**0.0437**
3	0.0610	0.0619	**0.1959**	**0.1944**	**0.0888**	**0.0879**
4	0.0149	0.0156	**0.1964**	**0.1954**	**0.1338**	**0.1325**
5	0.0029	0.0031	**0.1571**	**0.1571**	**0.1612**	**0.1598**
6	0.0005	0.0005	**0.1045**	**0.1052**	**0.1614**	**0.1606**
7	0.0001	0.0001	**0.0594**	**0.0604**	**0.1384**	**0.1384**
8	0.0000	0.0000	**0.0295**	**0.0304**	**0.1036**	**0.1043**
9	0.0000	0.0000	0.0130	0.0136	**0.0688**	**0.0699**
10	0.0000	0.0000	0.0051	0.0055	**0.0411**	**0.0421**
11	0.0000	0.0000	0.0018	0.0020	**0.0223**	**0.0231**
12	0.0000	0.0000	0.0006	0.0007	**0.0110**	**0.0116**
13	0.0000	0.0000	0.0002	0.0002	**0.0050**	**0.0054**
14	0.0000	0.0000	0.0001	0.0001	0.0021	0.0023
15	0.0000	0.0000	0.0000	0.0000	0.0008	0.0009
16	0.0000	0.0000	0.0000	0.0000	0.0003	0.0004
17	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1: Probability mass function $\mathrm{P}\{M_m = i\}$ for $\alpha = 0.99$ under the exact and approximate distribution to four decimal digits with acceptance regions highlighted.