Aggregate Risk

MFM Practitioner Module: Quantitative Risk Management

John Dodson

January 31, 2018

Introduction

As we discussed last semester, the general goal of risk measurement is to come up with a single metric that can be used to make financial risk management decisions. This is a difficult task; but axiomatic concepts such as –

- monotonicity
- translation-invariance
- law-invariance
- sub-additivity
- quasi-convexity
- coherence
- positive homogeneity

have proven to be useful in at least defining the problem. The notions of acceptance sets and the dual representation have also led to important insights.

Coherent Properties

Let us preview some possible qualities.

- money-equivalent $\varrho(L)$ and L are in the same units
- estimable $\varrho(L)$ is non-random
- constant $\mathrm{P}\{L = l\} = 1 \Rightarrow \varrho(L) = l$
- positive homogeneous $\lambda \geq 0 \Rightarrow \varrho(\lambda L) = \lambda\varrho(L)$
- translation invariant $\varrho(L + l) = \varrho(L) + l$ for const. l
- sub-additive $\varrho(L_1 + L_2) \leq \varrho(L_1) + \varrho(L_2)$
- co-monotonic additive $h(\cdot)$ invertible, increasing $L_2 = h(L_1) \Rightarrow \varrho(L_1 + L_2) = \varrho(L_1) + \varrho(L_2)$
- convex $0 \leq \lambda \leq 1 \Rightarrow$ $\varrho(\lambda L_1 + (1-\lambda)L_2) \leq \lambda\varrho(L_1) + (1-\lambda)\varrho(L_2)$
- risk aversion $\mathrm{E}\, L = 0 \Rightarrow \varrho(L + l) \geq l$ for const. l

Coherent Properties

Discussion

- Constancy, translation invariance, and risk aversion all refer to the role of cash in the portfolio.
- Risk aversion says that the investor should prefer cash to any risky portfolio with the same expected outcome.
- Sub-additivity and concavity both refer to the risk-reducing role of diversification.
- Co-monotonic additivity means that derivatives or leverage in the portfolio provide no diversification benefit relative to underlyings (for known implied vol).
- Positive homogeneity mean that the risk can be decomposed into the sum of marginal risks.

Coherent Properties

Quasi-Convexity

Any concave index of satisfaction is also quasi-convex, which means that

$$\varrho(\lambda L_1 + (1-\lambda)L_2) \leq \max(\varrho(L_1), \varrho(L_2)) \quad \forall \quad 0 < \lambda < 1$$

The converse is not true in general; but in fact quasi-convexity is sufficient to express the preference for diversification. To see this, consider an incumbent portfolio with loss L_1 and any candidate portfolio with loss L_2 with $\varrho(L_2) \leq \varrho(L_1)$ with quasi-convex risk $\varrho(\cdot)$. The definition leads us to conclude that for any $0 < \lambda < 1$, $\varrho(L_3) \leq \varrho(L_1)$ where

$$L_3 = \lambda L_1 + (1-\lambda)L_2$$

That is, no convex combination of the incumbent and candidate portfolio is riskier than the incumbent.

Acceptance Sets

An important observation about coherent risk measures is that they can always be expressed in terms of convex sets A_ϱ of acceptable loss outcomes.

$$\varrho(L) = \inf \{m \in \mathbb{R} : L - m \in A_\varrho\}$$

Clearly if ϱ is law-invariant, then A_ϱ depends on the risk measure $\mathbb{P}$.

Example: Value-at-Risk

Value-at-risk is a good example. It is monotone and translation-invariant, but not necessarily coherent. The acceptable set the set of losses such that $\mathrm{P}\left(L > 0\right) \leq 1 - \alpha$. Depending on $\mathbb{P}$, this may or may not be convex.

Insights from the theory of acceptance sets was important in the development of the principle of coherence.

Dual Representation

The notion of the dual representation of risk measures is an important theoretical insight which has subsequently influenced practical developments in risk measurement. It is similar to the notion of acceptance sets of losses, but instead we consider an acceptable set of probability measures.

Dual Representation for Coherent Risk Measures

If $\varrho(L)$ is coherent, there is a set of probability measures $\mathcal{Q}_\varrho$ for which

$$\varrho(L) = \max\left(\mathrm{E}^{\mathbb{Q}}\, L : \mathbb{Q} \in \mathcal{Q}_\varrho\right)$$

Example: Expected Shortfall

For expected shortfall, which is generally coherent, we have

$$\mathcal{Q}_\varrho = \left\{\mathbb{Q} : \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{1-\alpha}\right\}$$

in terms of the Radon-Nikodym derivative.

Distortion Metrics

Expected shortfall is the canonical example of a so-called distortion risk measure, which are defined in general as

$$\varrho(L) = \int_0^1 q_u(L)\, dD(u)$$

in terms of the loss quantile function $q_u(L) \triangleq F_L^{\leftarrow}(u)$ and a convex, increasing, absolutely continuous distortion function $D : [0,1] \mapsto [0,1]$ with $D(0) = 0$ and $D(1) = 1$. Distortion risk measures are coherent. They are also comonotone additive, i.e. invariant to leverage.

Example: Expected Shortfall

For expected shortfall, the distortion function is just

$$D(u) = (1-\alpha)^{-1}(u-\alpha)^+$$

All distortion risk measures can be expressed as an expectation of ES_α under some measure $\mu(\alpha)$.

Coherent Stress Tests

For a positive homogeneous risk measure and affine loss, we can write

$$\varrho(m + \boldsymbol{\lambda}'\boldsymbol{X}) = m + r_\varrho(\boldsymbol{\lambda})$$

If we define the set of risk factor scenarios S_ϱ as

$$S_\varrho = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{u}'\boldsymbol{x} \leq r_\varrho(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in \mathbb{R}^d \right\}$$

then

$$\varrho(L) = \sup \left\{ m + \boldsymbol{\lambda}'\boldsymbol{x} : \boldsymbol{x} \in S_\varrho \right\}$$

Note that the scenario set does not depend on the allocations $\boldsymbol{\lambda}$.
Since this is equivalent to a dual representation with $\mathcal{Q}_\varrho$ consisting of degenerate measures on the elements of S_ϱ, it is coherent.

Linear Loss and Elliptical Risk Factors

Suppose that the risk factors are an elliptic random vector, $\boldsymbol{X} \sim E_d\left(\boldsymbol{\mu}, \Sigma, \psi\right)$, and suppose that potential losses are affine in the risk factors, $L = m + \boldsymbol{\lambda}' X$. Then for any law-invariant, translation-invariant, positive homogeneous risk measure ϱ,

$$\varrho(L) = m + \boldsymbol{\lambda}'\boldsymbol{\mu} + \varrho(Y_1)\sqrt{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}}$$

where $\boldsymbol{Y} \sim S_d(\psi)$.
If furthermore $\varrho(Y_1) > 0$ and $\boldsymbol{X}$ has a finite covariance matrix, then ϱ is a sub-additive risk measure, and

$$\varrho(L) = \mathrm{E}\, L + k_\varrho \sqrt{\operatorname{var} L}$$

which means that ϱ is consistent with weak stochastic dominance and that optimal portfolios lie on the Markowitz mean-variance efficient frontier.

Euler Decomposition

A notable feature of the standard deviation of a weighted sum of correlated random variables is that it can be expressed as a weighted sum of partial standard deviations.

$$\varrho(L) = \varrho(\boldsymbol{\lambda}'\boldsymbol{L}) = r_\varrho(\boldsymbol{\lambda}) \triangleq \sqrt{\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda}} = \boldsymbol{\lambda}'\frac{\Sigma\boldsymbol{\lambda}}{r_\varrho(\boldsymbol{\lambda})}$$

This is true of any risk measures that are (first-order) positive homogeneous, which is to say any risk measure such that for $\forall\lambda \geq 0$, $\varrho(\lambda L) = \lambda\varrho(L)$, which is clearly true for expectation and standard deviation and less obviously true for value-at-risk and expected shortfall.

Allocation by Gradient

In the case where $\boldsymbol{\lambda} = \mathbf{1}$ hence $L = L_1 + L_2 + \cdots + L_d$, this means

$$\varrho(L) = \sum_{i=1}^{d} \frac{\partial r_\varrho}{\partial \lambda_i}(\mathbf{1}) \triangleq \sum_{i=1}^{d} AC_i^\varrho$$