Quantitative Risk Management
Case for Week 3 (Spring)

John A. Dodson

February 6, 2019

Let us consider the expected shortfall index of satisfaction for a very simple portfolio: λ shares in an asset whose value today is $p > 0$ and whose horizon value P is lognormal.

Let us assume that the objective measure is mark-to-market profit; therefore in the text's notation, we have (apologies for the signs)

$$\begin{aligned} -L &= \lambda (P - p) \\ &= \lambda p \left(e^X - 1\right) \end{aligned}$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean μ and variance $\Sigma > 0$. The risk measure is

$$-\varrho(L) = -r_\varrho(\lambda) = \frac{1}{1-c} \int_0^{1-c} F_{-L}^{\leftarrow}(q)\, dq$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$\begin{aligned} F_{-L}(z) &= \mathrm{P}\left\{-L < z\right\} \\ &= \mathrm{P}\left\{\lambda p \left(e^X - 1\right) < z\right\} \\ &= \mathrm{P}\left\{X \operatorname{sgn}\lambda < \log\left(1 + \frac{z}{\lambda p}\right) \operatorname{sgn}\lambda\right\} \\ &= \mathrm{P}\left\{\frac{X-\mu}{\sqrt{\Sigma}} \operatorname{sgn}\lambda < \frac{\log\left(1 + \frac{z}{\lambda p}\right) - \mu}{\sqrt{\Sigma}} \operatorname{sgn}\lambda\right\} \\ &= \Phi\left(\frac{\log\left(1 + \frac{z}{\lambda p}\right) - \mu}{\sqrt{\Sigma} \operatorname{sgn}\lambda}\right) \end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the distribution function, is therefore

$$F_{-L}^{\leftarrow}(q) = \lambda p \left(e^{\mu + \operatorname{sgn}\lambda \sqrt{\Sigma}\Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$\begin{aligned}
-r_{\varrho}(\lambda) &= \frac{1}{1-c}\int_0^{1-c} \lambda p \left(e^{\mu + \operatorname{sgn}\lambda \sqrt{\Sigma}\Phi^{-1}(q)} - 1 \right)\, dq \\
&= \lambda p \left(\frac{1}{1-c}\int_0^{1-c} e^{\mu + \operatorname{sgn}\lambda \sqrt{\Sigma}\Phi^{-1}(q)}\, dq - 1 \right) \\
&= \lambda p \left(\frac{1}{1-c}\int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \operatorname{sgn}\lambda \sqrt{\Sigma} z}\phi(z)\, dz - 1 \right)
\end{aligned}$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \operatorname{sgn}\lambda \sqrt{\Sigma} z}\phi(z) = e^{\mu + \frac{1}{2}\Sigma}\phi\left(z - \operatorname{sgn}\lambda\sqrt{\Sigma} \right)$$

we have the final result,

$$r_{\varrho}(\lambda) = -\lambda p \left(e^{\mu + \frac{1}{2}\Sigma}\frac{1}{1-c}\Phi\left(\Phi^{-1}(1-c) - \operatorname{sgn}\lambda\sqrt{\Sigma} \right) - 1 \right) \tag{1}$$

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$r_{\varrho}(\lambda) \approx -\lambda p \mu + \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c}|\lambda| p\sqrt{\Sigma} \tag{2}$$

which is in the form $\varrho(L) = \operatorname{E} L + k_{\varrho} \operatorname{std} L$ that we have seen before.

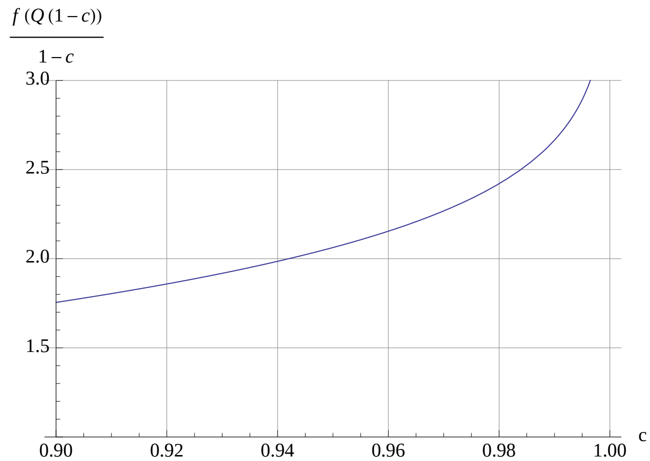
Let us spend a moment interpreting this. A long ($\lambda > 0$) is less risky if the asset has a positive expected return ($\mu > 0$), and a short ($\lambda < 0$) is less risky if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance increases risk for any non-zero position.

This all seems quite reasonable for a rational risk measure.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the loss. In a Delta-Gamma setting, we can replace the objective by the quadratic

$$-L = \lambda p \left(e^X - 1 \right) \approx \lambda p \left(X + \tfrac{1}{2}X^2 \right)$$



hence $\Theta_\lambda = 0$, $\Delta_\lambda = \lambda p$, and $\Gamma_\lambda = \lambda p$. Let us define a new objective[1] to represent this approximation.

$$\Xi_\lambda = \lambda p \left(X + \tfrac{1}{2} X^2 \right)$$

Is is straight-forward (but tedious) to work out that the first several central moments of this are

$$\begin{aligned}
\mathrm{E}\left(\Xi_\lambda\right) &= \lambda p \left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma \right) \\
\mathrm{Sd}\left(\Xi_\lambda\right) &= |\lambda| p \sqrt{\Sigma} \sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma} \\
\mathrm{Sk}\left(\Xi_\lambda\right) &= 3 \operatorname{sgn} \lambda \sqrt{\Sigma} \frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{\left((1+\mu)^2 + \tfrac{1}{2}\Sigma\right)^{3/2}}
\end{aligned}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$-r_\varrho(\lambda) \approx \mathrm{E}\left(\Xi_\lambda\right) + \mathrm{Sd}\left(\Xi_\lambda\right) \left(z_1 + \frac{z_2 - 1}{6} \mathrm{Sk}\left(\Xi_\lambda\right) \right)$$

with coefficients

$$\begin{aligned}
z_1 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)\, dq = -\frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \\
z_2 &= \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)^2\, dq = 1 - \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} \Phi^{-1}(1-c)
\end{aligned}$$

depending on the confidence level $c < 1$[2].

Putting this together, we get a third expression for the index of satisfaction.

$$\begin{aligned}
r_\varrho(\lambda) \approx -\lambda p \left(\mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\Sigma \right) &+ \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c} |\lambda| p \sqrt{\Sigma} \\
&\cdot \left(\sqrt{(1+\mu)^2 + \tfrac{1}{2}\Sigma} + \tfrac{1}{2} \operatorname{sgn} \lambda \frac{(1+\mu)^2 + \tfrac{1}{3}\Sigma}{(1+\mu)^2 + \tfrac{1}{2}\Sigma} \Phi^{-1}(1-c) \sqrt{\Sigma} \right)
\end{aligned} \tag{3}$$

This result agrees with (2) to lowest order in μ and $\sqrt{\Sigma}$.

[1]The objective random variable is the profit, which is the negative of the loss.

[2]The trick to these integrals is to realize that $\phi'(z) = -z\phi(z)$.

4 Modeling Default

Our horizon asset value P is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$-L' = \lambda p \left(Y e^{X} - 1 \right)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before[3], but now we add an independent default indicator $Y \sim \mathrm{Bern}(1 - h)$ for default probability h.

[3]Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.