Quantitative Risk Management
Spring Assignment

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This assignment is not a regular homework. It is an individual project worth half of the module grade for the spring term. If you discuss this assignment with anyone other than the instructor, please summarize those discussions in a statement acknowledging and describing your collaborations.

Please share your solution with me through Google Drive with a timestamp before 5:30 PM on Thursday, February 28[1]. Please turn in your report directly to me. You are welcome to discuss the project with our TA, but she will not be grading it.

Introduction

In the fourth homework this term we analyzed a stylized defaultable bond with tenor T, fixed continuous coupon k, constant default hazard rate γ, constant risk-free short rate r, and "recovery of Treasury" $1-\delta$ specified as a Bernoulli random variable whose state space is the set $\{0,1\}$ with $\mathrm{E}^{\mathbb{P}}[\delta]=L$.

We noted that, since the recovery is not hedgeable, the risk-free value of the bond is not uniquely defined; and some other mechanism is required to set the price at which buyers and sellers will trade.

For sake of argument, I asked you to calculate the $\mathbb{P}$-measure expected value of the bond price and the equivalent par yield. But rational agents will not pay full fair value for a risky bet. Taking this into account, the price from the last assignment must be too high. In this assignment, let's attempt to estimate this bias.

In the utility framework, the market clears if the price of the bond maximizes the expected value, $\mathrm{E}^{\mathbb{P}}[u(\Psi)]$, of some appropriate representative agent's utility with respect to some appropriate $\mathbb{P}$-measurable objective Ψ that integrates the partial equilibrium valuations.

Say the objective is the net asset value from holding the bond until $t=h$ in the limit $h\downarrow 0$ and the utility is isoelastic for $\psi>0$

$$u(\psi)=\frac{\psi^{1-\eta}-1}{1-\eta}$$

for constant relative risk aversion $\eta>0$.

Assume that the equilibrium bond price as an element of the set of all convex combinations of the two partial equilibrium values,

$$\{(1-\theta)p_0(r,\gamma)+\theta\, p_1(r,\gamma)\,:\,0\leq\theta\leq 1\}$$

*The original version specified a different objective and utility.

[1] The deadline was extended from February 27.

Problem Statement

Please produce a report or presentation explaining the problems below, **10 points each**, your solution approach, intermediate and final results, and potential interpretations.

1. Please include a solution to the previous assignment, deriving the partial equilibrium values for this stylized defaultable bond from first principles.
2. Continuing that narrative, explain the utility framework and find an expression for the the value of
$$\theta^\star = \arg\max_{\theta\in[0,1]} \mathrm{E}^{\mathbb{P}}\left[u\left(\Psi(\theta)\right)\right]$$
3. In terms of this optimal control, simplify the formula for the equilibrium value of the bond and the equivalent par yield, and describe any constraints on the relative risk aversion.
4. Plot the equivalent par yield versus the relatvive risk aversion (over its domain) for a stylized bond with $T = 10.$, $r = 0.$, $\gamma = 0.001$, and $L = 0.5$.

N.B.: In evaluating the expected value of the utility, you may assume that the hazard rate for the default process is the same under $\mathbb{P}$ and $\mathbb{Q}$. Also, You can assume that default occurs at $t = h$, so that the coupon income over the period provides a positive floor on the net asset value even in the event of default with no recovery.

Hints: In evaluating the state space for Ψ there are three potential sources for profit/loss to consider between $t = 0$ and $t = h$: income from the coupon payment, capital gain/loss from the reduction of the tenor, and the severity a potential default. The sample space has three elements corresponding to: status quo, default with full recovery, and default with no recovery. Remember to enforce a wealth constraint at $t = 0$.

Grading Rubric

Ten out of fifty points will be based on the follow criteria:

- You follow the instructions. **(2 points)**
- I can reproduce your results in the last question with the code and documentation you provide. **(2 points)**
- Your report or presentation is clear and professional, including citations and collaboration statements. **(6 points)**

Solution

The result from the previous assignment about the two risk-neutral values for the stylized bond, So, the defaultable bond value can be re-expressed as

$$p_\delta(r,\gamma) = \frac{k+\gamma-\delta\gamma}{r+\gamma} + \left(1 - \frac{k+\gamma-\delta\gamma}{r+\gamma}\right)e^{-(r+\gamma)T} \quad \text{for} \quad \delta\in\{0,1\}$$

so the convex combination (calling out the control variable and the tenor in this notation) is

$$p(\theta,T) = \frac{k+\gamma-\theta\gamma}{r+\gamma} + \left(1 - \frac{k+\gamma-\theta\gamma}{r+\gamma}\right)e^{-(r+\gamma)T} \quad \text{for} \quad 0\le\theta\le 1$$

Wealth Objective

The wealth at $t = 0$ does not depend on the control variable θ; but the value of the bond at $t = 0$ clearly does. Therefore the quantity owned at $t = 0$ must compensate. Say the initial wealth is one unit of account. Then the quantity of face held during the investment period must be $1/p(\theta, T)$.

Let's move on to evaluating the states of wealth at $t = h$, which we will term Ψ_h. We are only interested in small h, so let's assume that any default event happens exactly at $t = h$.

In all states the the bond generates income whose future value is

$$k\frac{e^{rh}-1}{r}\frac{1}{p(\theta,T)} \approx \frac{kh}{p(\theta,T)} \tag{income}$$

If it does not default and $p(\theta, T) \neq 1$ the bond produces some capital gain/loss as the premium amortizes.

$$\frac{p(\theta,T-h)}{p(\theta,T)} - 1 \approx -h\frac{\partial p(\theta,T)}{\partial T}\frac{1}{p(\theta,T)} = \left(r + \gamma - \frac{k+\gamma(1-\theta)}{p(\theta,T)}\right) h \tag{capital gains}$$

The gain/loss from a potential default event.

$$\frac{1-\delta}{p(\theta,T)} - 1 \tag{impairment}$$

We do not actually know the probability of default under $\mathbb{P}$. Under $\mathbb{Q}$ it is $1 - e^{-\gamma h} \approx \gamma h$. We will assume it is the same under $\mathbb{P}$. Generally there is no risk premium for uncertainty in γ because, unlike in the case of loss given default, whether the default probability over a certain period is known or simply has a known expected value outcomes are indistinguishable.

In summary, the horizon profit random variable is

$$\Psi_h(\theta) \approx \begin{cases} 1 + \left(r + \gamma\left(1 - \frac{1-\theta}{p(\theta,T)}\right)\right) h & \mathrm{P}^{\mathbb{P}}\{\tau > h\} \approx 1 - h\gamma \\ \frac{1+kh}{p(\theta,T)} & \mathrm{P}^{\mathbb{P}}\{\tau \le h \;\&\; \delta = 0\} \approx h\gamma\,(1-L) \\ \frac{kh}{p(\theta,T)} & \mathrm{P}^{\mathbb{P}}\{\tau \le h \;\&\; \delta = 1\} \approx h\gamma\, L \end{cases}$$

for small $h > 0$.

Expected Utility

We have been able to express the states and probabilities for the objective as first-order in h. The expected value of the utility is a convex combination of terms of the form $u(a + bh)$,

$$\mathrm{E}^{\mathbb{P}}\left[u\left(\Psi_h(\theta)\right)\right] \approx u\left(1 + b_1(\theta)h\right)\cdot(1 - c_1 h) + u\left(a_2(\theta) + b_3(\theta)h\right)\cdot(c_1 - c_3)h + u\left(b_3(\theta)h\right)\cdot c_3 h$$

Since we want to continue to evaluate this to lowest order in h, note that

$$u(a + bh) \approx u(a) + a^{-\eta} b\, h \quad \text{for} \quad a > 0$$

for small $h > 0$, and that $u(1) = 0$. So this is no zero-order term in the expected value of the utility. There are several contributors to a first-order term; but notice that the default with no recovery includes a term with $h^{2-\eta}$, whose power is smaller than one for $\eta > 1$.

$$\mathrm{E}^{\mathbb{P}}\left[u\left(\Psi_h(\theta)\right)\right] \approx \frac{b_3(\theta)^{1-\eta}}{1-\eta} c_3\, h^{2-\eta} + \left(b_1(\theta) + \frac{a_2(\theta)^{1-\eta}(c_1 - c_3) - c_1}{1-\eta}\right) h$$

Optimality

This suggests a few different regimes in the relative risk aversion η: For $\eta > 2$, the expected utility does not have a finite expectation for $h \downarrow 0$ at all; for $1 < \eta \leq 2$, the first term is dominant; and for $0 < \eta < 1$, the second term is dominant[2].

Let's handle the $1 < \eta \leq 2$ case first. Since $b_3(\theta) = k/p(\theta, T)$, the optimization problem in the limit $h \downarrow 0$ is

$$\theta^\star \in \arg\min_{0 \leq \theta \leq 1} p(\theta, T)^{\eta - 1} \qquad (\, 1 < \eta \leq 2 \,)$$

which is equivalent to just minimizing $p(\theta, T)$, so the constraint binds and the solution is trivially $\theta^\star = 1$ for which $p\left(\theta^\star, T\right) = p_1(r, \gamma)$. The par yield is just $y = r + \gamma$.

But this is deceptive. Since the expected utility is negative for $\eta > 1$, the representative agent would actually rather hold interest-free cash than the bond if given the choice. So this solution is actually just an artifact of the wealth constraint.

In summary, there does not seem to be a realistic solution for $\eta > 1$.

For $0 < \eta < 1$,

$$\mathrm{E}^{\mathbb{P}}\left[u\left(\Psi_h(\theta)\right)\right] \approx \left(r + \gamma\left(1 - \frac{1-\theta}{p(\theta, T)} + \frac{p(\theta, T)^{\eta-1}(1-L) - 1}{1-\eta}\right)\right) h$$

which is maximized when

$$\theta^\star \in \arg\min_{0 \leq \theta \leq 1} \frac{1-\theta}{p(\theta, T)} - \frac{1-L}{(1-\eta) p(\theta, T)^{1-\eta}} \qquad (\, 0 < \eta < 1 \,)$$

Since $p(\theta, T)$ is linear θ, the first-order condition in $\theta^\star$ is not difficult to evaluate:

$$0 = -\frac{p(1, T) + (1-L)\left(p(0, T) - p(1, T)\right) p(\theta^\star, T)^\eta}{p(\theta^\star, T)^2}$$

Since $p(0, T) > p(1, T)$, the RHS above is always negative and there is no solution on the domain on θ, so as in the case above with the constraint binds and $\theta^\star = 1$.

Par Yield

To get the par yield, we must first determine the par coupon, which is the value of $k^\star$ which solves

$$1 = \frac{k^\star + \gamma - \theta^\star \gamma}{r + \gamma} + \left(1 - \frac{k^\star + \gamma - \theta^\star \gamma}{r + \gamma}\right) e^{-(r+\gamma)T}$$

or, since $1 - e^{-(r+\gamma)T} > 0$,

$$k^\star = r + \theta^\star \gamma \qquad \text{(par coupon)}$$

The bond yield is the internal rate of return that equates the value of the defaultable bond with the value of a default-free version of the bond. In particular for a par bond, this internal rate of return must equal the coupon rate, so

$$y = k^\star = r + \theta^\star \gamma \qquad \text{(par yield)}$$

[2] For $\eta = 1$ this degenerates into logarithmic utility.