Let us consider the expected shortfall index of satisfaction for a very simple portfolio: $\lambda$ shares in an asset whose value today is $p > 0$ and whose horizon value $P$ is lognormal.

Let us assume that the objective measure is mark-to-market profit; therefore in the text’s notation, we have (apologies for the signs)

\[-L = \lambda (P - p) = \lambda p (e^X - 1)\]

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean $\mu$ and variance $\Sigma > 0$. The risk measure is

\[-\varrho(L) = -\varrho(\lambda) = \frac{1}{1-c} \int_0^{1-c} F_{\varrho}^{-1}(q) \, dq\]

for confidence level $c < 1$ in terms of the quantile function for the objective value.

**Exact Version**

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

\[F_{-L}(z) = P \{ -L < z \} = P \{ \lambda p (e^X - 1) < z \} = P \left\{ X \text{ sgn} \lambda < \log \left( 1 + \frac{z}{\lambda p} \right) \text{ sgn} \lambda \right\} = P \left\{ \frac{X - \mu}{\sqrt{\Sigma}} \text{ sgn} \lambda < \frac{\log \left( 1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma}} \text{ sgn} \lambda \right\} = \Phi \left( \frac{\log \left( 1 + \frac{z}{\lambda p} \right) - \mu}{\sqrt{\Sigma} \text{ sgn} \lambda} \right)\]

where $\Phi(\cdot)$ is the CDF of a standard normal.
The quantile, which is the inverse of the distribution function, is therefore

\[ F^{-1}_{\text{L}}(q) = \lambda p \left( e^{\mu + \text{sgn} \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) \]

So can proceed to evaluate the index of satisfaction.

\[ -r_\varrho(\lambda) = \frac{1}{1 - c} \int_0^{1-c} \lambda p \left( e^{\mu + \text{sgn} \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq \]

\[ = \lambda p \left( \frac{1}{1 - c} \int_0^{1-c} e^{\mu + \text{sgn} \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right) \]

\[ = \lambda p \left( \frac{1}{1 - c} \int_{\Phi^{-1}(1-c)}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn} \sqrt{\Sigma} \phi(z)} dz - 1 \right) \]

where the last line is achieved by the change of variable \( z = \Phi^{-1}(q) \) and \( \phi(z) = \Phi'(z) \) is the density of a standard normal. Since

\[ e^{\mu + \text{sgn} \sqrt{\Sigma} \phi(z)} = e^{\mu + \frac{1}{2} \Sigma \phi} \left( z - \text{sgn} \lambda \sqrt{\Sigma} \right) \]

we have the final result,

\[ r_\varrho(\lambda) = -\lambda p \left( e^{\mu + \frac{1}{2} \Sigma} \frac{1}{1 - c} \Phi \left( \Phi^{-1}(1 - c) - \text{sgn} \lambda \sqrt{\Sigma} \right) - 1 \right) \]  

(1)

**Short Horizon Approximation**

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

\[ r_\varrho(\lambda) \approx -\lambda p \mu + \frac{\phi \left( \Phi^{-1}(1 - c) \right)}{1 - c} |\lambda| p \sqrt{\Sigma} \]  

which is in the form \( \varrho(L) = E L + k_\varrho \text{ std } L \) that we have seen before.

Let us spend a moment interpreting this. A long (\( \lambda > 0 \)) is less risky if the asset has a positive expected return (\( \mu > 0 \)), and a short (\( \lambda < 0 \)) is less risky if the asset has a negative expected return (\( \mu < 0 \)). In contrast, positive variance increases risk for any non-zero position.

This all seems quite reasonable for a rational risk measure.

**Cornish-Fisher Approximation**

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the loss. In a Delta-Gamma setting, we can replace the objective by the quadratic

\[ -L = \lambda p \left( e^{X} - 1 \right) \approx \lambda p \left( X + \frac{1}{2} X^2 \right) \]
hence $\lambda = 0$, $\Delta = \lambda p$, and $\Gamma = \lambda p$. Let us define a new objective\(^1\) to represent this approximation.

$$\Xi = \lambda p \left( X + \frac{1}{2}X^2 \right)$$

It is straightforward (but tedious) to work out that the first several central moments of this are

$$E(\Xi) = \lambda p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right)$$

$$Sd(\Xi) = \sqrt{\lambda p \nu \sqrt{V} \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma}}$$

$$Sk(\Xi) = 3 \text{sgn} \sqrt{\lambda \nu \sqrt{V}} \frac{(1 + \mu)^2 + \frac{1}{2} \Sigma}{\left( (1 + \mu)^2 + \frac{1}{2} \Sigma \right)^{3/2}}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$-r_{\phi}(\lambda) \approx E(\Xi) + Sd(\Xi) \left( z_1 + \frac{z_2 - 1}{6} Sk(\Xi) \right)$$

with coefficients

$$z_1 = \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q) \, dq = -\frac{\phi(\Phi^{-1}(1-c))}{1 - c}$$

$$z_2 = \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q)^2 \, dq = 1 - \frac{\phi(\Phi^{-1}(1-c))}{1 - c} \Phi^{-1}(1-c)$$

depending on the confidence level $c < 1^2$.\(^2\)

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\(^1\)The objective random variable is the profit, which is the negative of the loss.

\(^2\)The trick to these integrals is to realize that $\phi'(z) = -z \phi(z)$. 

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**Figure 1:** Factor for Delta-Gamma expected shortfall
Putting this together, we get a third expression for the index of satisfaction.

\[ r_e(\lambda) \approx -\lambda p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) + \frac{\phi \left( \Phi^{-1}(1 - c) \right)}{1 - c} |\lambda| p \sqrt{\Sigma} \]

\[ \cdot \left( \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \text{sgn} \lambda \left( \frac{(1 + \mu)^2 + \frac{1}{2} \Sigma}{(1 + \mu)^2 + \frac{1}{2} \Sigma} \Phi^{-1}(1 - c) \sqrt{\Sigma} \right) \right) \] (3)

This result agrees with (2) to lowest order in \( \mu \) and \( \sqrt{\Sigma} \).