

Quantitative Risk Management

Case for Week 5

John Dodson

November 25, 2020

Background

The identification of the non-degenerate normalized n -block maxima with the Fréchet, Gumbel, or Weibull random variables is the collective product of a number of probabilists working in Europe from the 1920's to the 1940's. For a survey of the important proofs, see [2].

This was a foundational result that helped define theoretical statistics and probability as a distinct sub-field of mathematics.

Today it is referred to as the “first extreme value theorem”. It became the “first” theorem in the 1970's after an extension, the “second” theorem, was discovered in relation to a panoply of contemporaneous experimental “power-law” results about the distributions of quantities in nature. I find this an interesting modern example of experimental science leading to mathematical developments.

The Second Extreme Value Theorem

The second extreme value theorem theorem says, in brief, that the relative probabilities of sufficiently extreme i.i.d. observations are arbitrarily similar to the relative probabilities of a generalized Pareto random variable.

I provide an informal demonstration of this connection below. Please see [1] for a more careful treatment.

Let's start with the excess distribution. If a random variable X has distribution $F(\cdot)$, the “excess distribution” is

$$\begin{aligned} F_\eta(x) &\triangleq \mathbb{P}[X - \eta \leq x | X > \eta] \\ &= \frac{F(x + \eta) - F(\eta)}{1 - F(\eta)} \\ &= 1 - \frac{1 - F(x + \eta)}{1 - F(\eta)} \end{aligned}$$

Let's further assume that the normalized n -block maxima converges to a Fréchet random variable (the other cases are similar). This means that for some sufficiently large n ,

$$F^n(c_n x + b_n) \approx \exp\left(-(1 + \xi x)^{-1/\xi}\right)$$

or

$$\begin{aligned} F(x) &\approx \exp\left(-\frac{1}{n}\left(1 + \xi \frac{x - b_n}{c_n}\right)^{-1/\xi}\right) \\ &\approx 1 - \frac{1}{n}\left(1 + \xi \frac{x - b_n}{c_n}\right)^{-1/\xi} \end{aligned}$$

where we are using the approximation $e^{-x} \approx 1 - x$ for small x in the second line. Substituting this into the excess distribution, we get

$$\begin{aligned} F_\eta(x) &\approx 1 - \frac{\left(1 + \xi \frac{x + \eta - b_n}{c_n}\right)^{-1/\xi}}{\left(1 + \xi \frac{\eta - b_n}{c_n}\right)^{-1/\xi}} \\ &= 1 - \left(1 + \xi \frac{x}{c_n + \xi(\eta - b_n)}\right)^{-1/\xi} \end{aligned}$$

So, defining $\beta(\eta) \triangleq c_n + \xi(\eta - b_n)$, which notably does not depend on x , we get the result

$$F_\eta(x) \approx 1 - \left(1 + \xi \frac{x}{\beta(\eta)}\right)^{-1/\xi}$$

which says that the excess is approximated by a generalized Pareto random variable.

References

- [1] August A. Balkema and Laurens de Haan. Residual life time at great age. *The Annals of Probability*, 2:792–804, 1974.
- [2] Boris V. Gnedenko. On the limit distribution of the maximum term of a random series. (in French) *Annals of Mathematics*, 44:423–453, 1943.