Quantitative Risk Management
Case for Week 6

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This case uses a classic result in portfolio theory to demonstrate an application of (and inspiration for) factor models.

Let us examine the equilibrium allocation under a profit objective, exponential utility, and normal markets.

Investment Problem

For a portfolio with static holdings[1] $\alpha_0 \mathbin{+\!\!+} \alpha$ (α_0 represents cash[2], $\cdot \mathbin{+\!\!+} \cdot$ represents concatenation) with net asset value

$$w = \alpha_0 + \alpha^\intercal p \tag{1}$$

the profit over τ years is

$$\Psi = \alpha_0 r \tau + \alpha^\intercal M \tag{2}$$

which crucially is linear[3] in the "market vector"

$$M = P - p \tag{3}$$

with P the random variable for asset prices, including any cashflows, $\tau > 0$ years in the future, and r the (simple-interest) rate of return on cash. The lower-case p represents the current prices, of course.

Let us assume that the market vector is normal,

$$M \sim \mathcal{N}\left(\mu\,\tau, \Sigma\,\tau\right) \tag{4}$$

and that the preferences of the representative agent are described by exponential utility

$$u(\psi) = \zeta\left(1 - e^{-\frac{\psi}{\zeta}}\right) \tag{5}$$

with absolute risk aversion $1/\zeta > 0$.

[1] Holdings are static in shares, not necessarily in weights.

[2] Cash is held out because its future value, expressed as a random variable, is degenerate.

[3] This argument fails for objectives based on compound returns.

Optimality

Let us consider the portfolios that satisfy a wealth constraint $w^\star$ and maximize expected utility.

$$\operatorname{E} u(\Psi) = \zeta \left(1 - e^{-\frac{\alpha_0}{\zeta} r\tau} \operatorname{E} e^{-\frac{\alpha^\intercal}{\zeta} M}\right) \tag{6}$$

$$= \zeta \left(1 - e^{-\frac{w^\star - \alpha^\intercal p}{\zeta} r\tau - \frac{\alpha^\intercal}{\zeta}\mu\tau + \frac{1}{2}\frac{\alpha^\intercal}{\zeta}\Sigma\tau\frac{\alpha}{\zeta}}\right) \tag{7}$$

In particular, the "certainty-equivalent" profit for this portfolio is

$$u^{-1}\left(\operatorname{E} u(\Psi)\right) = \left(w^\star r + \alpha^\intercal(\mu - pr) - \frac{1}{2\zeta}\alpha^\intercal\Sigma\alpha\right)\tau \tag{8}$$

It is apparent that an optimal portfolio satisfies

$$\alpha^\star \in \arg\max_{\alpha} \alpha^\intercal(\mu - pr) - \frac{1}{2\zeta}\alpha^\intercal\Sigma\alpha \tag{9}$$

If the covariance of the market vector is positive-definite ($\alpha^\intercal\Sigma\alpha > 0 \ \forall\alpha$) the first-order condition on the optimal portfolio is

$$\mu = pr + \frac{1}{\zeta}\Sigma\alpha^\star \tag{10}$$

whose unique solution is

$$\boxed{\alpha^\star = \zeta\Sigma^{-1}(\mu - pr)} \tag{11}$$

The "risk premium" is the discount to the certainty-equivalent rate of return that the representative agent would accept to eliminate uncertainty. For the optimal portfolio, this is

$$\frac{u^{-1}\left(\operatorname{E} u\left(\Psi^\star\right)\right)}{w^\star\tau} - r = \frac{\alpha^{\star\intercal}\Sigma\alpha^\star}{2\zeta w^\star} \tag{12}$$

There is an extensive empirical literature around the risk premium for U.S. investors. A typical result is that it is around 7% per year. If investors can expect an annual "volatility rate", $\sqrt{\alpha^{\star\intercal}\Sigma\alpha^\star}/w^\star$, of around 20%, that implies that the representative agent with wealth $w^\star$ has a risk aversion of about

$$\frac{1}{\zeta} \approx \frac{3.5}{w^\star}$$

Discussion

Notice that

$$\operatorname{E}\Psi^\star = w^\star r\tau + \frac{1}{\zeta}\operatorname{var}\Psi^\star \tag{13}$$

and more generally that

$$\operatorname{E}\Psi = wr\tau + \frac{1}{\zeta}\operatorname{cov}\left(\Psi, \Psi^\star\right) \tag{14}$$

$$= wr\tau + \frac{\operatorname{cov}\left(\Psi, \Psi^\star\right)}{\operatorname{var}\Psi^\star}\left(\operatorname{E}\Psi^\star - w^\star r\tau\right) \tag{15}$$

This is more recognizable to a student of finance when expressed in terms of rates of return:

$$\mathrm{E}\,\frac{\Psi}{w\tau} = r + \frac{\operatorname{cov}\left(\frac{\Psi}{w\tau}, \frac{\Psi^\star}{w^\star\tau}\right)}{\operatorname{var}\frac{\Psi^\star}{w^\star\tau}} \left(\mathrm{E}\,\frac{\Psi^\star}{w^\star\tau} - r\right) \tag{16}$$

where the coefficient is akin to the portfolio "beta" of the capital asset pricing model (CAPM), the correlation with the "market portfolio" times the ratio of the standard deviations of the rates of return.

Factor Model

If we assume that: (i) the model is broadly correct; (ii) the holdings and allocations of the optimal portfolio are observable; (iii) the representative agent's risk aversion is observable; and (iv) asset volatilities and correlations can be estimated precisely, we can use the result of the model to constrain the market model.

Consider a portfolio consisting of a single share of the i-th stock.

$$\frac{\Psi}{w} = \frac{P_i}{p_i} - 1 \tag{17}$$

Hence

$$\mathrm{E}\,P_i = p_i(1 + r\tau) + \lambda \operatorname{cor}(P_i, \Psi^\star)\sqrt{\tau \operatorname{var} P_i} \tag{18}$$

where

$$\boxed{\lambda \triangleq \frac{\sqrt{\alpha^{\star\intercal}\Sigma\alpha^\star}}{\zeta}} \tag{19}$$

with dimensions $\mathrm{yr}^{-1/2}$ is termed the "market price of risk" and notably depends on neither the asset nor the investment horizon[4].

In particular, the expected value of the (simple) return on the i-th asset is

$$\bar{R}_i \triangleq r + \lambda \operatorname{cor}(P_i, \Psi^\star)\sqrt{\frac{\operatorname{var} P_i}{p_i^2 \tau}} \tag{20}$$

whereby

$$\mathrm{E}\,P_i = p_i\left(1 + \bar{R}_i\tau\right) \tag{21}$$

[4] For the numerical values above, $\lambda \approx 0.7\,\mathrm{yr}^{-1/2}$.