Quantitative Risk Management
Case for Week 4

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October 4, 2021

Background

The identification of the non-degenerate normalized n-block maxima with the Fréchet, Gumbel, or Weibull random variables is the collective product of a number of probabilists working in Europe from the 1920's to the 1940's. For a survey of the important proofs, see [2].

This was a foundational result that helped cement theoretical statistics and probability as a distinct subfield of mathematics.

Today it is referred to as the "first extreme value theorem". It became the "first" theorem in the 1970's after an extension, the "second" theorem, was discovered in relation to a panoply of contemporaneous experimental "power-law" results about the distributions of quantities in nature.

I find this an interesting modern example of new mathematics arising from experimental science.

The Second Extreme Value Theorem

The second extreme value theorem theorem says, in brief, that the relative probabilities of sufficiently extreme i.i.d. observations are arbitrarily similar to the relative probabilities of a generalized Pareto random variable.

I provide an informal demonstration of this connection below. Please see [1] for a more careful treatment.

Let's start with the excess distribution. If a random variable X has distribution $F(\cdot)$, the **excess distribution** is

$$\begin{aligned} F_\eta(x) &= \mathrm{P}\left(X - \eta \leq x | X > \eta\right) \\ &= \frac{F(x+\eta) - F(\eta)}{1 - F(\eta)} \\ &= 1 - \frac{1 - F(x+\eta)}{1 - F(\eta)} \end{aligned}$$

Let's further assume, per the first extreme value theorem, that the normalized n-block maxima converges to a Fréchet random variable (the other cases are similar). That means that there exist sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that, for some sufficiently large n,

$$F^n\left(c_n x + b_n\right) \approx \exp\left(-(1+\xi x)^{-1/\xi}\right)$$

or

$$
\begin{aligned}
F(x) &\approx \exp\left(-\frac{1}{n}\left(1+\xi\frac{x-b_n}{c_n}\right)^{-1/\xi}\right) \\
&\approx 1-\frac{1}{n}\left(1+\xi\frac{x-b_n}{c_n}\right)^{-1/\xi}
\end{aligned}
$$

where we are using the approximation $e^{-x} \approx 1-x$ for small x in the second line.

Substituting this into the excess distribution, we get

$$
\begin{aligned}
F_\eta(x) &\approx 1-\frac{\left(1+\xi\frac{x+\eta-b_n}{c_n}\right)^{-1/\xi}}{\left(1+\xi\frac{\eta-b_n}{c_n}\right)^{-1/\xi}} \\
&= 1-\left(1+\xi\frac{x}{c_n+\xi(\eta-b_n)}\right)^{-1/\xi}
\end{aligned}
$$

So, defining $\beta(\eta)=c_n+\xi(\eta-b_n)$, which notably *does not* depend on x, we get the result

$$
F_\eta(x) \approx 1-\left(1+\xi\frac{x}{\beta(\eta)}\right)^{-1/\xi}
$$

which says that the excess is approximated by a generalized Pareto random variable.

Note that the generalized Pareto is self-similar, in the sense that excess distributions for a given random variable X at various sufficiently large thresholds η, η' have the same tail parameter ξ and have scale parameters related by $\beta' = \beta + \xi(\eta' - \eta)$. Therefore, it is only necessary to estimate the generalized Pareto parameters at a single threshold.

References

[1] August A. Balkema and Laurens de Haan. Residual life time at great age. *The Annals of Probability*, 2:792–804, 1974.

[2] Boris V. Gnedenko. On the limit distribution of the maximum term of a random series. (in French) *Annals of Mathematics*, 44:423–453, 1943.