Quantitative Risk Management
Case for Week 5

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This case uses a classic result in portfolio theory to demonstrate an application of (and inspiration for) factor models.

Investment Problem

Let us examine the equilibrium allocation under a profit objective, exponential utility, and normal markets.

For a portfolio with static holdings[1] $\alpha_0 \mathbin{+\!\!+} \alpha$ (α_0 represents cash[2], $\cdot \mathbin{+\!\!+} \cdot$ represents concatenation) with net asset value

$$w = \alpha_0 + \alpha^\intercal p \tag{1}$$

the profit over τ years is

$$\Psi = \alpha_0 r \tau + \alpha^\intercal M \tag{2}$$

which, crucially, is linear[3] in the **market vector**

$$M = P - p \tag{3}$$

with P the random variable for asset prices, including any cashflows, $\tau > 0$ years in the future, and r the (simple-interest) rate of return on cash. The lower-case p represents the current prices, of course.

Let us assume that the market vector is normal,

$$M \sim \mathcal{N}\left(\mu\,\tau, \Sigma\,\tau\right) \tag{4}$$

and that the preferences of the representative agent are described by exponential utility

$$u(\psi) = \zeta\left(1 - e^{-\frac{\psi}{\zeta}}\right) \tag{5}$$

with absolute risk aversion $1/\zeta > 0$.

[1] Holdings are static in shares, not necessarily in weights.

[2] Cash is held out because its future value, expressed as a random variable, is degenerate.

[3] This argument fails for objectives based on compound returns.

Optimality

Let us consider the portfolios that satisfy a wealth constraint $w^\star$ and maximize expected utility.

$$\mathrm{E}\, u(\Psi) = \zeta \left(1 - e^{-\frac{\alpha_0}{\zeta} r\tau}\, \mathrm{E}\, e^{-\frac{\alpha^\intercal}{\zeta} M}\right) \tag{6}$$

$$= \zeta \left(1 - e^{-\frac{w^\star - \alpha^\intercal p}{\zeta} r\tau - \frac{\alpha^\intercal}{\zeta}\mu\tau + \frac{1}{2}\frac{\alpha^\intercal}{\zeta}\Sigma\tau\frac{\alpha}{\zeta}}\right) \tag{7}$$

In particular, the **certainty-equivalent profit** for this portfolio is

$$u^{-1}\left(\mathrm{E}\, u(\Psi)\right) = \left(w^\star r + \alpha^\intercal(\mu - pr) - \frac{1}{2\zeta}\alpha^\intercal\Sigma\alpha\right)\tau \tag{8}$$

It is apparent that an optimal portfolio satisfies

$$\alpha^\star \in \arg\max_{\alpha} \alpha^\intercal(\mu - pr) - \frac{1}{2\zeta}\alpha^\intercal\Sigma\alpha \tag{9}$$

If the covariance of the market vector is positive-definite ($\alpha^\intercal\Sigma\alpha > 0\ \forall\alpha$) the first-order condition on the optimal portfolio is

$$\mu = pr + \frac{1}{\zeta}\Sigma\alpha^\star \tag{10}$$

whose unique solution is

$$\boxed{\alpha^\star = \zeta\Sigma^{-1}(\mu - pr)} \tag{11}$$

The **risk premium** is the discount to the certainty-equivalent rate of return that the representative agent would accept to eliminate uncertainty. For the optimal portfolio, this is

$$\frac{u^{-1}\left(\mathrm{E}\, u\left(\Psi^\star\right)\right)}{w^\star\tau} - r = \frac{\alpha^{\star\intercal}\Sigma\alpha^\star}{2\zeta w^\star} \tag{12}$$

There is an extensive empirical literature around the risk premium for U.S. investors. A typical result is that it is around 7% per year. If investors can expect an annual **volatility rate**, $\sqrt{\alpha^{\star\intercal}\Sigma\alpha^\star}/w^\star$, of around 20%, that implies that the representative agent with wealth $w^\star$ has a risk aversion of about

$$\frac{1}{\zeta} \approx \frac{3.5}{w^\star}$$

Discussion

Notice that

$$\mathrm{E}\,\Psi^\star = w^\star r\tau + \frac{1}{\zeta}\,\mathrm{var}\,\Psi^\star \tag{13}$$

and more generally that

$$\mathrm{E}\,\Psi = wr\tau + \frac{1}{\zeta}\,\mathrm{cov}\left(\Psi, \Psi^\star\right) \tag{14}$$

$$= wr\tau + \frac{\mathrm{cov}\left(\Psi, \Psi^\star\right)}{\mathrm{var}\,\Psi^\star}\left(\mathrm{E}\,\Psi^\star - w^\star r\tau\right) \tag{15}$$

This is more recognizable to a student of finance when expressed in terms of rates of return:

$$\mathrm{E}\,\frac{\Psi}{w\tau} = r + \frac{\mathrm{cov}\left(\frac{\Psi}{w\tau}, \frac{\Psi^\star}{w^\star\tau}\right)}{\mathrm{var}\,\frac{\Psi^\star}{w^\star\tau}}\left(\mathrm{E}\,\frac{\Psi^\star}{w^\star\tau} - r\right) \tag{16}$$

where the coefficient is akin to the portfolio **beta** of the capital asset pricing model ("CAPM"), the correlation with the **market portfolio** times the ratio of the standard deviations of the rates of return.

Factor Model

If we assume that: (i) the model is broadly correct; (ii) the holdings and allocations of the optimal portfolio are observable; (iii) the representative agent's risk aversion is observable; and (iv) asset volatilities and correlations can be estimated precisely, we can use the result of the model to constrain the market model.

Consider a portfolio consisting of a single share of the i-th stock.

$$\frac{\Psi}{w} = \frac{P_i}{p_i} - 1 \tag{17}$$

Hence

$$\mathrm{E}\,P_i = p_i(1 + r\tau) + \lambda\,\mathrm{cor}\,(P_i, \Psi^\star)\,\sqrt{\tau\,\mathrm{var}\,P_i} \tag{18}$$

where

$$\boxed{\lambda = \frac{\sqrt{\alpha^{\star\intercal}\Sigma\alpha^\star}}{\zeta}} \tag{19}$$

with dimensions $\mathrm{yr}^{-1/2}$ is termed the **market price of risk** and notably depends on neither the asset nor the investment horizon[4].

In particular, the expected value of the (simple) return on the i-th asset is

$$\bar{R}_i = r + \lambda\,\mathrm{cor}\,(P_i, \Psi^\star)\sqrt{\frac{\mathrm{var}\,P_i}{p_i^2\tau}} \tag{20}$$

whereby

$$\mathrm{E}\,P_i = p_i\left(1 + \bar{R}_i\tau\right) \tag{21}$$

[4]For the numerical values above, $\lambda \approx 0.7\,\mathrm{yr}^{-1/2}$.