Quantitative Risk Management
Homework for Week 1

John Dodson

September 13, 2021

Solutions to these problems are due at the beginning of the next session, which is 5:30 PM on Monday, September 20. Please turn in your solutions to the TA. Include your U of Minn. student identification number on your submission to facilitate recording marks in the Canvas learning management system. Also include the names of any classmates you consulted with in developing your solutions.

Problems

1. In each of the following situations a certain notion of capital discussed in QRM §2.1.3 is most relevant for the decision maker. Explain which notion of capital that is. **(2 points)**
   (a) a financial analyst who uses balance sheet data to value a firm
   (b) a chief risk officer of an insurance company who has to decide on the appropriate level of reinsurance
   (c) a regulator who has to decide on shutting down a bank with many bad loans on its book
2. Let L be a random loss and $h : \mathbb{R} \to \mathbb{R}$ a continuous and strictly increasing function. Show that **(3 points)**
$$\mathsf{VaR}_\alpha\left(h(L)\right) = h\left(\mathsf{VaR}_\alpha(L)\right) \quad \forall \alpha \in (0,1)$$
3. Consider two stocks whose log-returns are bivariate normal with annualized volatilities of $\sigma_1 = 0.20$ and $\sigma_2 = 0.25$ and correlation $\rho = 0.4$. Assume that the expected returns are zero and that one year consists of 252 trading days. Consider a portfolio with current value $V_t = 1{,}000{,}000.$ (in USD) and portfolio weights $w_1 = 0.7$ and $w_2 = 0.3$. Furthermore, denote by L^Δ_{t+1} the linearized loss (using log-prices of the stocks as risk factors). Compute the daily $\mathsf{VaR}_{0.99}\left(L^\Delta_{t+1}\right)$ and $\mathsf{ES}_{0.99}\left(L^\Delta_{t+1}\right)$ for the portfolio. How does the answer change if $\rho = 0.6$? **(5 points)**

Solutions

1. Our text presents three three broad concepts of capital in Chapter 2: equity capital, regulatory capital, and economic capital.

   Equity capital is based on financial accounting standards, and it is the most readily observable version of capital for a public company. In the US, public companies are required to file financial statements quarterly with the Securities and Exchange Commission, which makes them publicly available through the EDGAR system. Some firms, such as broker-dealers, are also subject to non-public *monthly* financial reporting through the FINRA eFOCUS system. **A financial analyst could reasonably use a company's EDGAR reports to measure equity capital.** A credit officer at a self-regulatory organization, such as a clearinghouse, could do something similar with the more frequent FOCUS reports.

   Regulators can oblige firms to measure and report capital based on information that is not normally available in financial reports, such as the composition of assets or liabilities or even a portfolio's value-at-risk or other risk metric. Regulators know that effective regulation is transparent and principals-based; so the methodology for calculating regulatory capital is usually based on assumptions that have universal applicability even if they are not optimal for any particular firm. **If a regulator determines to shut down a troubled bank, it will almost certainty cite an adverse change in the bank's regulatory capital (or stress test results) in the public narrative.**

   Economic capital is analogous to managerial accounting: It is optimized to a firm's management's perception of risk (including the risk of being perceived as under-capitalized[1]) and is generally not intended to be part of any pubic discourse. **The CRO of an insurance company will generally justify the purchase of reinsurance policies to the firm's board by arguing that the savings in the cost of maintaining economic capital is greater than the expense of the policies.**

2. Let $h_{\text{min}} = \lim_{x\downarrow-\infty} h(x)$, which might be $-\infty$. h^{-1} is a non-decreasing function, so $h(x) > y \Leftrightarrow x > h^{-1}(y)$ for all $y > h_{\text{min}}$. From definition (2.8),

$$\begin{aligned}
\mathsf{VaR}_\alpha\left(h(L)\right) &= \inf\{l \in \mathbb{R} : \mathrm{P}\left[h(L) > l\right] \leq 1-\alpha\} \\
&= \inf\{l > h_{\text{min}} : \mathrm{P}\left[h(L) > l\right] \leq 1-\alpha\} \\
&= \inf\left\{l > h_{\text{min}} : \mathrm{P}\left[L > h^{-1}(l)\right] \leq 1-\alpha\right\} \\
&= h\left(\inf\left\{h^{-1}(l) \in \mathbb{R} : \mathrm{P}\left[L > h^{-1}(l)\right] \leq 1-\alpha\right\}\right) \\
&= h\left(\inf\{l \in \mathbb{R} : \mathrm{P}\left[L > l\right] \leq 1-\alpha\}\right) \\
&= h\left(\mathsf{VaR}_\alpha(L)\right)
\end{aligned}$$

   for all $0 < \alpha < 1$, where the fourth assertion of equality is a consequence of the continuity of h.

   This result facilitates the proof that value-at-risk is a *comonotone additive* risk measure, which will come up again in Chapter 8. The significance of this is to limit the ability of derivatives or leverage to diversify risk.

3. Let $X = (X_1, X_2)'$ be a random (column) vector of the log-returns on the two stocks between t and $t+1$, which is a bivariate normal with covariance matrix

$$\Sigma = \frac{1}{252}\begin{pmatrix} 0.20 \times 0.20 & 0.25 \times 0.20 \times 0.4 \\ 0.20 \times 0.25 \times 0.4 & 0.25 \times 0.25 \end{pmatrix}$$

[1] The 2012 "London Whale" debacle at JPMorgan Chase was a result of management's attempt to decrease its regulatory capital requirement by offsetting its credit exposure from commercial banking activity with over-the-counter credit index derivatives.

Note that since daily returns are independent, and (co)variances of sums of independent variables are just sums of (co)variances, we can "de-annualize" the volatility by scaling its square by the number of "days" in the year.

The loss, $L_{t+1} = V_t - V_{t+1}$, is

$$L_{t+1} = 700{,}000.\left(1 - e^{X_1}\right) + 300{,}000.\left(1 - e^{X_2}\right)$$

which is linearized as

$$L^{\Delta}_{t+1} = -700{,}000.\, X_1 - 300{,}000.\, X_2 = b' X \quad \text{where} \quad b = \begin{pmatrix} -700{,}000. \\ -300{,}000. \end{pmatrix}$$

Since the marginals of X are normals, the linearized loss is also a (univariate) normal random variable. Normal random variables are parameterized by mean and variance. The mean is zero by assumption; and the variance is

$$\text{var}\left[L^{\Delta}_{t+1}\right] = b'\,\Sigma\, b \approx (11{,}551.30)^2$$

Since for a standard normal random variable $Z \sim \mathcal{N}(0,1)$, $\text{P}[Z > 2.3263] = \Phi(-2.3263) \approx 1 - 0.99$, and since $L^{\Delta}_{t+1}/11{,}551.30$ is approximately a standard normal random variable,

$$\boxed{\mathsf{VaR}_{0.99}\left(L^{\Delta}_{t+1}\right) \approx 2.3263 \times 11{,}551.30 \approx 26{,}872.34}$$

While the value-at-risk is a quantile, the expected shortfall is a conditional expectation. A conditional expectation is just an expectation with respect to a conditional probability density; and a conditional probability density is just an unconditional probability density, restricted to some sub-algebra of events, re-scaled by the measure of the restriction ($1 - \alpha$ in this case).

Denote the density of a standard normal r.v. by $\phi(z) = \Phi'(z)$ and note that $\phi'(z) = -z\,\phi(z)$. The expected shortfall at confidence α is

$$\begin{aligned} \text{E}\left[Z | Z > \Phi^{-1}(\alpha)\right] &= \int_{\Phi^{-1}(\alpha)}^{\infty} z\, \frac{\phi(z)}{1-\alpha}\, dz \\ &= \int_{\Phi^{-1}(\alpha)}^{\infty} \frac{-\phi'(z)}{1-\alpha}\, dz \\ &= \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \end{aligned}$$

So, for $\alpha = 0.99$, $\text{E}\left[Z | Z > 2.3263\right] \approx 2.6652$. Since our loss has a different scale,

$$\boxed{\mathsf{ES}_{0.99}\left(L^{\Delta}_{t+1}\right) \approx 2.6652 \times 11{,}551.30 \approx 30{,}786.69}$$

For the higher correlation, the variance of the loss is also a little higher, and the one-day 99% value-at-risk and expected shortfall evaluate to $\boxed{28{,}501.25}$ and $\boxed{32{,}652.87}$ respectively.