Quantitative Risk Management
Homework for Week 4

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October 4, 2021

Solutions to these problems are due at the beginning of the next session, which is 5:30 PM on Monday, October 11. Please turn in your solutions to the TA. Include your U of Minn. student identification number on your submission to facilitate recording marks in the Canvas learning management system. Also include the names of any classmates you consulted with in developing your solutions.

Deep Out-of-the-Money Options

We should be able to use the results of extreme value theory to value sufficiently deep OTM European-style options (low-strike puts and high-strike calls). This is convenient, because Black-Scholes implied volatility tends to diverge in this case.

Say there exists lower and upper thresholds, η_L and η_U, such that the risk-neutral distribution of S_T is partially characterized by

$$F(x) = \mathrm{P}(S_T < x) = \begin{cases} 0 & x \le \eta_L + \frac{\beta_L}{\xi_L} \\ \theta_L \left(1 - \xi_L \frac{x-\eta_L}{\beta_L}\right)^{-1/\xi_L} & \eta_L + \frac{\beta_L}{\xi_L} < x \le \eta_L \\ ? & \eta_L < x \le \eta_U \\ 1 - \theta_U \left(1 + \xi_U \frac{x-\eta_U}{\beta_U}\right)^{-1/\xi_U} & x > \eta_U \end{cases}$$

for some tail parameters ξ_L, ξ_U with $\xi_L \le -\frac{\beta_L}{\eta_L}$ and $\xi_U \ge 0$ (for support), positive scales β_L, β_U, and positive weights θ_L, θ_U with $\theta_L + \theta_U < 1$ (for monotonicity). Note that the question mark represents the region of the state space for which the Pareto approximation does not apply.

Based on arbitrage pricing theory and put/call parity, we can write the value of a European-style put with strike price K and expiration date T as

$$\begin{aligned} p(T,K) &= d(T)\,\mathrm{E}\left[\max(K - S_T, 0)\right] && (K \le \eta_L) && (1) \\ &= d(T)\left(\mathrm{E}\left[\max(S_T - K, 0)\right] + K - f(T)\right) && (K \ge \eta_U) && (2) \end{aligned}$$

with discount factor $d(T)$ and underlying forward price $f(T)$.

Problems

1. Evaluate the expectations in (1) and (2) for $K \le \eta_L$ and $K \ge \eta_U$. **(2 points each)**

2. Using the 2-September prices we have for the 17-September NASDAQ-100 options, fit a single set of values for θ_L, β_L, ξ_L to puts with strikes of 11000 and below. **(3 points)**

3. Using the same data, fit a single set of values for θ_U, β_U, ξ_U for strikes of 17000 and above. **(3 points)**

Assumptions

Assume that $\eta_L = 11000$ and $\eta_U = 17000$.

Suggestions

To fit parameters, I suggest you build an objective function and apply an optimization algorithm such as Nelder-Mead. Do this separately for the upper and lower parameters.

Depending on your implementation of Nelder-Mead, you can implement constraints (such as positivity) by having your objective function return `NaN` when evaluating out-of-bounds arguments. The algorithm should recognize that as a signal to search in a different direction.

A smooth objective, such as mean squared error (MSE) can be easier to optimize. But it is also sensitive to outliers. Mean absolute error is more robust but may be harder to optimize. You may need to manually omit outliers and use MSE.

When using a local algorithm, such as Nelder-Mead, that requires initialization, check several starting points to convince yourself that the local minimum you get is also the global minimum.

Solution

Let us start with the low strikes. Assume $\eta_L + \frac{\beta_L}{\xi_L} < K \leq \eta_L$.

$$E\left[\max(K - S_T, 0)\right] = \int_{\eta_L + \frac{\beta_L}{\xi_L}}^{K} (K - x) F'(x)\, dx$$

It is easy enough to take the derivative in the integrand and evaluate the indefinite integral. But it is more efficient to to start by applying integration by parts, viz. $(K - x) \cdot F(x)\big|_{\eta_L + \frac{\beta_L}{\xi_L}}^{K} = 0$:

$$\ldots = \int_{\eta_L + \frac{\beta_L}{\xi_L}}^{K} F(x)\, dx$$

With the obvious change of variables we get

$$\begin{aligned} &= \theta_L \frac{\beta_L}{-\xi_L} \int_0^{1 - \xi_L \frac{K - \eta_L}{\beta_L}} u^{-1/\xi_L}\, du \\ &= \theta_L \frac{\beta_L}{1 - \xi_L} \left(1 - \xi_L \frac{K - \eta_L}{\beta_L}\right)^{1 - 1/\xi_L} \end{aligned}$$

Similarly for $K > \eta_U$ (assuming $0 < \xi_U < 1$) with integration by parts based on $(K-x)\cdot(1-F(x))|_K^\infty = 0$,

$$\begin{aligned}
\mathrm{E}\left[\max(S_T - K, 0)\right] &= \int_K^\infty (x-K)F'(x)\,dx \\
&= \int_K^\infty \left(1 - F(x)\right)\,dx \\
&= \theta_U \frac{\beta_U}{\xi_U} \int_{1+\xi_U \frac{K-\eta_U}{\beta_U}}^\infty u^{-1/\xi_U}\,du \\
&= \theta_U \frac{\beta_U}{1-\xi_U} \left(1 + \xi_U \frac{K-\eta_U}{\beta_U}\right)^{1-1/\xi_U}
\end{aligned}$$

For $\xi_U = 0$, take the limit to get $\theta_U \beta_U e^{-(K-\eta_U)/\beta_U}$. For $\xi_U \geq 1$ the integral diverges.

Since the results are similar for the lower and upper tails, I implemented a single function to evaluate this.

```
"""forward time value for a deep out-of-the money European-style options
valued using the asymptotic Generalized Pareto"""
function dotm(K,η,θ,β,ξ)
    if (ξ < 0 && ξ > -β/η) || ξ ≥ 1 || β ≤ 0 || θ ≤ 0
        return NaN
    elseif ξ < 0 && K < η+β/ξ
        return 0.
    elseif ξ > 0 && ξ < eps()
        return θ*β*exp(-(K-η)/β)
    else
        return θ*β/(1-ξ)*(1+abs(ξ)*(K-η)/β)^(1-1/ξ)
    end
end
```

In my script, the put and strike prices are in the vectors `puts` and `strike` and the discount and forward are in `disc` and `fwd`.

The forward option time-values for the low and high strikes, which are the target for the model fit, can be collected as:

```
dataL = puts[strikes.≤ηL]/disc
dataU = puts[strikes.≥ηU]/disc-strikes[strikes.≥ηU].+fwd
```

and the corresponding model values are:

```
modelL = params -> map(K->dotm(K,ηL,params...),strikes[strikes.≤ηL])
modelU = params -> map(K->dotm(K,ηU,params...),strikes[strikes.≥ηU])
```

Note that the `...` operator converts the parameter vector $(\theta, \beta, \xi)'$ into a sequence of function arguments.

I found that both the MSE and the MAE objectives were manageable. I present results with the MSE version. The objective functions to be minimized are

```
objL = params -> sum((modelL(params)-dataL).^2)
objU = params -> sum((modelU(params)-dataU).^2)
```

If we employ a local algorithm for minimizing the objective function to obtain estimates for the parameter values, we should have some sense of where to begin the search.

Our data are likely to be the most reliable for strikes near the thresholds. Conveniently, there is a simple expression for the expectations above right at the threshold, and an even simpler expression for the slope of this expectation with respect to the strike price.

	model	data
$\mathrm{E}\left[\max(K - S_T, 0)\right]\big\|_{K=\eta_L}$	$\theta_L \frac{\beta_L}{1-\xi_L}$	1.0
$\frac{d}{dK}\,\mathrm{E}\left[\max(K - S_T, 0)\right]\big\|_{K=\eta_L}$	θ_L	0.0012
$\mathrm{E}\left[\max(S_T - K, 0)\right]\big\|_{K=\eta_U}$	$\theta_U \frac{\beta_U}{1-\xi_U}$	1.2
$\frac{d}{dK}\,\mathrm{E}\left[\max(S_T - K, 0)\right]\big\|_{K=\eta_U}$	$-\theta_U$	-0.0024

If we assume that the values for ξ_L and ξ_U are both close to zero, these results suggest initializing the parameter search near $\theta_L \approx 0.0012$, $\beta_L \approx 830$, $\theta_U \approx 0.0024$, and $\beta_U \approx 500$. Note that the upper bound on ξ_L, which is $-\beta_L/\eta_L$, is about -0.08 under these values.

The results from `Optim.optimize`,

```
resL = optimize(objL,[0.0012,830.,-0.1])
resU = optimize(objU,[0.0024,500.,0.1])
```

are tabulated below.

	lower	upper
η	11000	17000
$\hat{\theta}$	0.00129	0.00268
$\hat{\beta}$	1850.	468.
$\hat{\xi}$	-1.20	1.64×10^{-10}

Table 1: Generalized Pareto parameters for the risk-neutral distribution on the September 17, 2021, opening price of the NASDAQ-100 based on index options at the close of September 2.

The resulting fits are plotted on the next page.

Discussion

It is interesting that the upper tail seems to be so close to Gumbel. It might be worth exploring whether this is a stylized fact of index options and if there is some economic justification. Of course there are outliers for the very high strikes; but they do not seem to be too influential. The ones with strikes above 18000 look like they might not be synchronized.

The lower tail is very curious. It seems as if the market on 2-September is putting essentially no price on 19-September index values below about 9450, or about 40% below the forward. Obviously a 40% decline in the NASDAQ-100 index in under a month would be an unprecedented global economic event; and it is hard to imagine it without some exogenous cause such as a sudden war or natural disaster. Why would the market chose to assign no value at all to such scenarios? Perhaps the market doubts the clearinghouse's ability to pay out in such a scenario. Or perhaps this result is just an artifact of the option time value being a similar magnitude as the bid/offer spread.

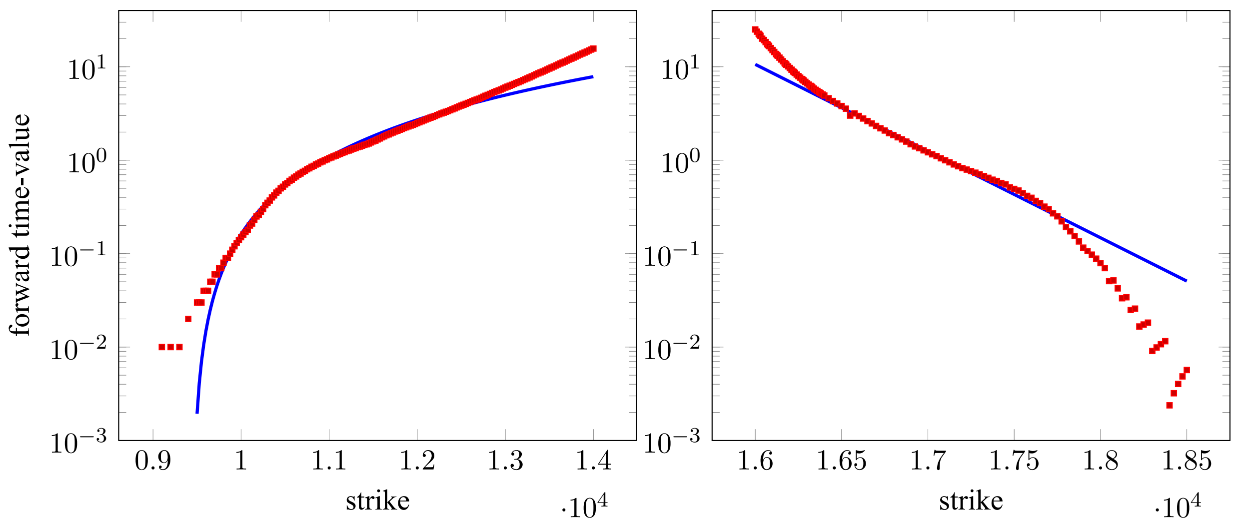

Figure 1: Tails of 17-September NASDAQ-100 index option time-values on 2-September versus GP model fits.

Note that we are modeling only a tiny portion of the state space here: only a few parts per thousand under the risk-neutral measure. And we certainly cannot generalize from an analysis of a single expiration on a single day.