Quantitative Risk Management
Homework for Week 5

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Solutions to these problems are due at the beginning of the next session, which is 5:30 PM on Monday, October 18. Please turn in your solutions to the TA. Include your U of Minn. student identification number on your submission to facilitate recording marks in the Canvas learning management system. Also include the names of any classmates you consulted with in developing your solutions.

Much has been written about the principal component analysis of yield curves. In this exercise, you will perform a principal component analysis of the U. S. Treasury bond market.

Problem

Bond yield is an important comparative metric to investors, but unlike the interbank loan/reserve/repo market, the Treasury market quotes and trades not with a yield convention but with a (clean[1]) price convention. Our analysis will be based on bond price innovations.

It is convenient to work with aggregate data rather than historical data on individual bonds. In particular, the Federal Reserve publishes a daily report of "constant maturity" aggregate yields, which is available at `https://www.federalreserve.gov/datadownload/Choose.aspx?rel=H15`. This is useful as a basis for timeseries invariants, allowing us to abstract away the details of individual bonds.

Active managers of Treasury securities typically finance[2] their trades. There is a liquid two-way market in Treasury repo to facilitate this. The New York Federal Reserve publishes a daily index of Treasury collateral overnight financing rates[3] based on this activity, which is described at `https://apps.newyorkfed.org/markets/autorates/sofr`.

Acquire & pre-process serial/panel data

1. Retrieve 1001 days (about four years) of daily data on 1–, 2–, 3–, 5–, 7–, 10–, 20–, and 30–year (nominal) constant maturity yields (in percent per annum[4]) through October 8, 2021, and convert these into log-return timeseries, which will be our invariants. **(1 point)**

To do this conversion, use the approximation from the Appendix:

$$\log\left[\frac{P_{t+\Delta t}}{P_t}\right] \approx y\,\Delta t - \frac{1-\left(1+\frac{y}{2}\right)^{-2\tau}}{y}\Delta y \tag{1}$$

[1] adjusted for the accrual of the current coupon

[2] buy bond – post bond as loan collateral – use proceeds (plus a little) to pay for the bond

[3] Treasury repo rates are quoted in basis points per annum.

[4] divide by 100 to get an annual rate

where y is annual yield rate, t is measured in years, and τ is the bond term (in years).

Adjust for serial hetero-skedasticity

2. Retrieve one thousand days of daily Treasury financing rates from the NY Fed website. Interpreting these as (annualized) conditional means, extract timeseries of daily residuals from the invariants. **(1 point)**

3. Use quasi maximum likelihood to fit GARCH(1,1) to the log-returns and use the conditional variances to standardize the residuals. **(2 points)**

Estimate the panel dispersion

Fixed income assets experience materially leptokurtotic (fat-tailed) innovations. Here we are interested in (co)variance, so we need an estimator that is robust to leptokurtosis, such as the M-estimator implemented in algorithm 6.29.

4. Use the M-estimator with $\nu = 4$ to estimate the term structure dispersion of the log-returns of Treasury bond prices. **(3 points)**

Extract the principal components

5. Calculate the eigenvalues of the dispersion matrix. The established concept of **bond duration** is predicated on an assumption of the prominence of systematic risk in (long-only) bond portfolios. How prominent is the largest eigenvector of the dispersion here? If you wanted to build a factor model for the Treasury market, how many factors would you use? **(3 points)**

Hint: The sum of the eigenvalues should be equal to the trace of the matrix, which is (approximately) eight in this case.

Solution

The H.15 and SOFR reports have 1001 daily constant maturity yield and financing rate observations on these `d = 8` series between October 11, 2017 and October 8, 2021, so `n = 1000` (business) daily excess returns. Note that there are typically between 249 and 251 banking days per year, with 250 days per year being a reasonable average[5].

The standard H.15 download comes as a CSV file and includes eleven different rates indexed by a ”Time Period” which in the daily version is just a single date. Weekends are excluded, but banking holidays are indicated by “ND” and unavailable rates are left blank. If you drop all missing rows there is continuous data from July 31, 2001.

```
df = dropmissing!(CSV.read("FRB_H15.csv",DataFrame,header=6,missingstrings=["","ND"]))
```

The last eight columns are the CMT yields we are interested in.

Normally, I would prefer to keep my data in a DataFrame. But since we need to perform matrix calculations for the panel analysis, I went ahead and extracted the daily returns into a dense matrix.

[5] In the U.S., stock exchanges and banks observe slightly different holidays.

```
"CMT tenors"
τ = [1,2,3,5,7,10,20,30]
"daily log-returns"
x = zeros(n,d)
Δt = 1/250
for i = 1:d
    cmt = df[end-n:end,4+i]/100
    y = cmt[1:end-1]
    Δy = diff(cmt)
    D = (1 .-(1 .+y/2).^(-2τ[i]))./y
    x[:,i] = y*Δt.-D.*Δy
end
```

Similarly, I ingested the CSV of secured funding rates and adjusted the daily log-returns to get residuals.

```
df2 = CSV.read("SOFR.csv",DataFrame)
"daily financing per unit"
μ = df2.rate[1:end-1]/10000*Δt*365.25/360
"residuals"
ε = x.-μ
```

Note that SOFR is an annual (basis point) rate with a ACT/360 daycount and the average number of "actual" days per interval is about $365.25/250 \approx 1.46$.

I used the module from the case a few weeks ago to standardize these residuals.

```
"standardized residuals"
z = zeros(n,d)
"parameters"
θ = zeros(3,d)
for i = 1:d
    θ[:,i] = garch_fit(ε[:,i])
    z[:,i] = ε[:,i]./sqrt.(garch(ε[:,i],θ[:,i]))
end
```

tenor	$1-\alpha_1-\beta_1$	α_1	β_1	σ_∞^2
1Y	0.008	0.128	0.864	0.06×10^{-6}
2Y	0.010	0.151	0.838	0.51×10^{-6}
3Y	0.018	0.127	0.854	1.09×10^{-6}
5Y	0.047	0.102	0.851	3.49×10^{-6}
7Y	0.071	0.095	0.835	7.23×10^{-6}
10Y	0.081	0.111	0.808	14.28×10^{-6}
20Y	0.076	0.124	0.800	50.51×10^{-6}
30Y	0.052	0.130	0.818	93.28×10^{-6}

Table 1: GARCH(1,1) parameters.

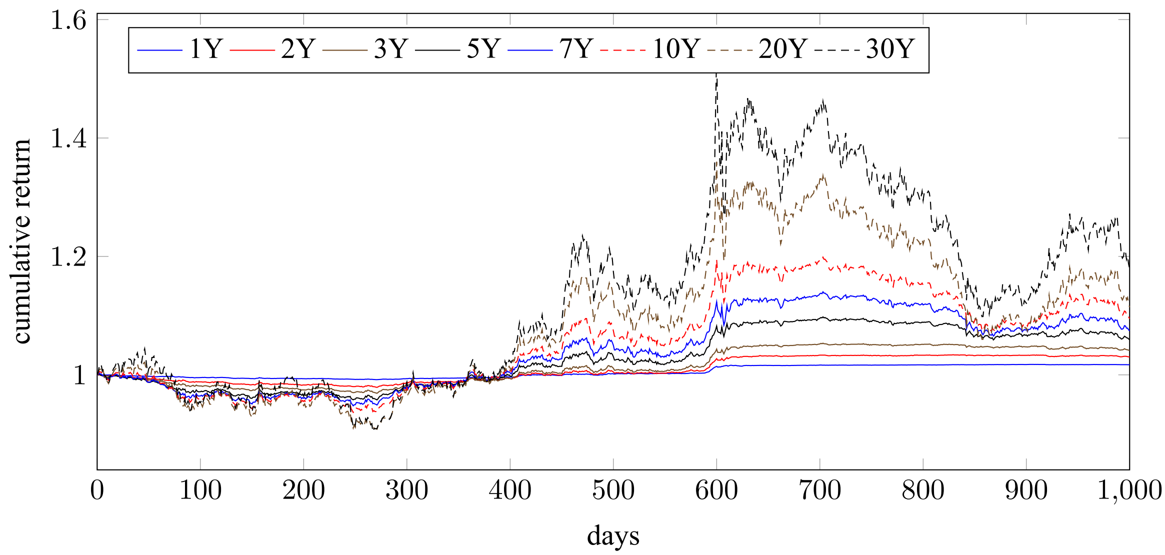

Figure 1: Cumulative excess return on constant-maturity Treasuries.

While the GARCH parameters are fairly consistent across CMT tenors, the magnitudes of the variance of the daily log-returns is vastly different. This is a well-known stylized fact of bonds hinting at a highly systemic structure. There is a heuristic measure for the sensitivity of a bond to the latent common factor: bond duration. Duration is the (negative) derivative of the function relating a bond's price and its yield. Under an assumption[6] that yield innovations are equal across tenors, duration measures the relative magnitudes of corresponding bond price innovations.

Now we can begin to look at the dependence. Since our data seems to be non-normal, the usual sample estimator for covariance is not appropriate. M-estimators allow us to estimate the dispersion of an elliptical random variable given the characteristic generator. Just as GARCH de-volatization homogenizes the data along the serial axis, the M-estimator imposes customized weights based on the degree of panel dispersion. From a process standpoint, it is filtering out jumps to focus on the diffusion.

```
"location/dispersion M-estimator, based on Student's-t"
function M_est(x,ν;iter=30)
    n,d = size(x)
    μ = fill(NaN,d)
    Σ = fill(NaN,n,n)
    w = ones(n)
    N = iter
    while N > 0
        μ = x'*w/sum(w)
        Σ = sum([w[i]*(x[i,:].-μ)*(x[i,:].-μ)' for i=1:n])/n
        invΣ = inv(Σ)
        w = [(ν+d)/(ν+(x[i,:].-μ)'*invΣ*(x[i,:].-μ)) for i=1:n]
        N -= 1
```

[6]A dynamical model in which yield innovations are strictly equal across tenors cannot be made arbitrage-free.

```
    end
    return μ,Σ
end

"robust location & dispersion estimate"
loc,disp = M_est(z,4)

"principal components"
pca = eigen(disp)
pca.values
```

The eigenvalues I got are in Table 2. The ratio of the largest to the total is **79%**.

λ_1	4.47
λ_2	0.64
λ_3	0.36
λ_4	0.12
λ_5	0.05
λ_6	0.02
λ_7	0.02
λ_8	0.01

Table 2: Eigenvalues of the standardized dispersion of the excess returns

The sum of the top three eigenvalues accounts for about 96% of the trace. This is a typical result, and it can be used to justify a low-dimensional latent factor model interpretation of risk in the Treasuries market. The first three factors are typically stylized as "shift", "tilt", and "twist", in response to diagrams such as Figure 2. This pattern is typical for term structures, and is a consequence of yields being generally smooth in tenor[7].

For inspiration, consider the case of a dispersion matrix in Topelitz form: where the diagonal elements and each of the off-diagonals elements are common; i.e.,

$$\Sigma = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & & \\ a_1 & a_0 & a_1 & a_2 & \dots & \\ a_2 & a_1 & a_0 & a_1 & a_2 & \dots \\ \vdots & \vdots & & \ddots & & \end{pmatrix}$$

For a Topelitz matrix the eigenvectors are proportional to $(b_1^i, b_2^i, b_3^i, \dots)'$ for $i = 0, 1, 2, \dots$. This means that, in particular, the first eigenvector is proportional to $(1, 1, 1, \dots)$.

[7]Discontinuities in interest rate term structures are not uncommon in the first year of tenor, due to tax and financial reporting considerations.

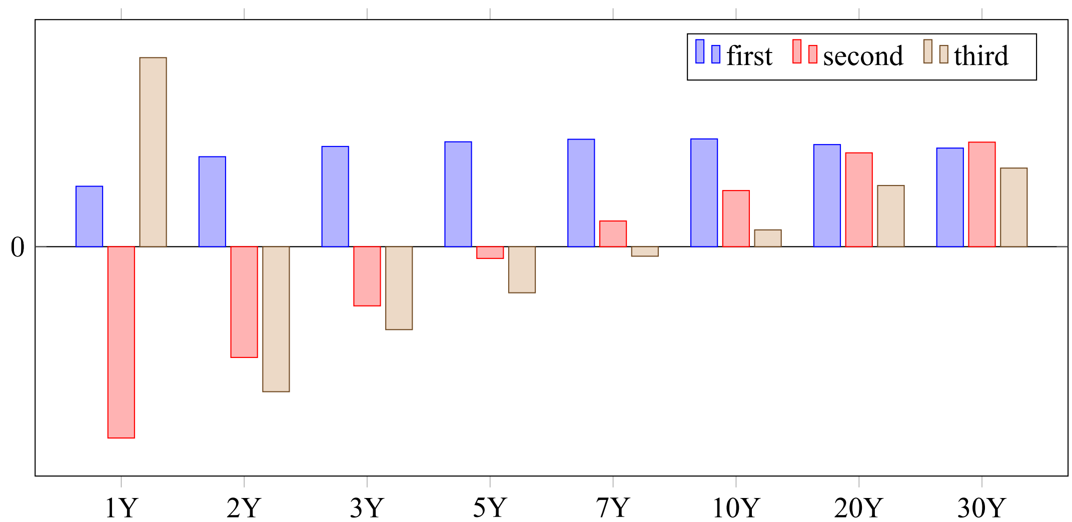

Figure 2: First three eigenvectors of the dispersion of the standardized daily excess returns of Treasury Bonds

Appendix

Yield

Yield is not return. It is not a performance measure, and (unlike stock returns) it usually has a materially non-zero (and non-constant) conditional mean over short horizons. Yield is a static concept that practitioners traditionally use to compare different bonds at an instant in time. Return, in contrast, is a measure of the performance of a single asset over time.

Let's translate a timeseries of bond yields into a timeseries of returns.

Often when we have a yield timeseries, we do not also have full information about the identity or characteristics of the particular bond from which the yield was extracted. So we cannot easily reverse engineer the corresponding price timeseries. In fact, sometimes the reported yield has been interpolated from several bonds; so in fact there is no precise bond. The Federal Reserve's H.15 report is an example of this. This data consists of "constant maturity" yields from the U. S. Treasuries market at various tenors: two years, five years, ten years, etc.

Let's consider a typical Treasury bond making equal semi-annual interest payments. Let the maturity date be T and the annual coupon rate c. The relationship between price and yield is given by

$$P_t = \frac{c}{y_t}\left(1 + \frac{y_t}{2}\right)^{2(t-t_0)} + \left(1 - \frac{c}{y_t}\right)\left(1 + \frac{y_t}{2}\right)^{-2(T-t+t_0)} \tag{2}$$

where t_0 is six months before the next coupon after t. This representation assumes coupon payments are equally spaced. The Treasury's yield calculation is slightly different, but these differences are immaterial—especially considering that the H.15 data is rounded to the nearest basis point (10^{-4} per year).

Note that if $c \approx y_{t_0}$, then $P_{t_0} \approx 1$; that is to say, the bond was issued at par. This is a good approximation for most bonds, whose coupons are typically set with this goal in mind, for example through an auction process.

Return

Let's interpret the yield in the next period: firstly as the new yield on the previous period's par bond, $c = y_t$; and secondly as the coupon on the next period's par bond. The value of the former bond is

$$P_{t+\Delta t} = \left(1 + \frac{y_t + \Delta y_{t+\Delta t}}{2}\right)^{2\Delta t} \frac{1 + \frac{\Delta y_{t+\Delta t}}{y_t}\left(1 + \frac{y_t + \Delta y_{t+\Delta t}}{2}\right)^{-2(T-t)}}{1 + \frac{\Delta y_{t+\Delta t}}{y_t}} \tag{3}$$

Since the initial value was one, the log-return on this bond is just the log of $P_{t+\Delta t}$. While this is simple enough to calculate exactly, a first-order approximation is very accurate

$$\log\left[\frac{P_{t+\Delta t}}{P_t}\right] \approx y_t\,\Delta t - \frac{1 - \left(1 + \frac{y_t}{2}\right)^{-2(T-t)}}{y_t}\Delta y_{t+\Delta t}$$

Notice that this approximation includes a carry term proportional to the yield and an exposure term proportional to the duration.