#1: In order to find the Cartesian equation for a plane we need to know a vector $\vec{n} = (A, B, C)$ which is perpendicular (or *normal*) to the plane, and a point (x_0, y_0, z_0) which is on the plane.

We can get a normal vector by taking the cross product of two vectors in the plane (technically, *parallel* to the plane). For example,

$$\vec{PQ} = (1, 1, 1)$$
$$\vec{PR} = (2, 0, 3)$$

You can check the following calculation on your own:

$$\vec{n} = \vec{PQ} \times \vec{PR} = (3, -1, -2)$$

Using the point $P(-1, 2, 0)$, the equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ for the plane is:

$$3(x + 1) - 1(y - 2) - 2(z - 0) = 0$$

You could also multiply this out to get an equation of the form $Ax + By + Cz = D$.

#2: The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is the ellipsoid with x-intercepts at $(\pm a, 0, 0)$, y-intercepts at $(0, \pm b, 0)$, and z-intercepts at $(0, 0, \pm c)$. Placing the north and south poles on the z-axis gives us

$$\frac{x^2}{3822^2} + \frac{y^2}{3822^2} + \frac{z^2}{3810^2} = 1$$

#3: $T : \mathbb{R}^3 \to \mathbb{R}^4$, so the matrix representing it should be a 4×3 matrix. It is as follows:

$$\begin{bmatrix} 0 & 2 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

The composition $T \circ S$ does not exist, because you would do S first, and get 4 outputs, but T requires 3 inputs.

The other composition, $S \circ T$, does exist. We can find it using matrices; multiply the matrix representing S (which you first have to find) and the matrix representing T:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

This is the matrix which represents the linear transformation

$$S(T(x,y,z)) = (z-y, -3x+y, x+y+z, 2y)$$

You could also do this without matrices by noting that S flip-flops its first and last inputs, and leaves everything else the same. Note that our final answer is exactly that, if the inputs happen to be the outputs of T (which is the case when computing $S \circ T$).

#4: Any points in the intersection must have x and y values on the circle of radius $\sqrt{3}$ described by the equation $x^2 + y^2 = 3$. The standard paramtrization for this circle is $(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$, where $0 \leq \theta \leq 2\pi$. We also know that any points on the intersection have z-values such that $z = 4x - 5y + 12$. Hence our parametrization is:

$$f(t) = (\sqrt{3}\cos\theta, \sqrt{3}\sin\theta, 4(\sqrt{3}\cos\theta) - 5(\sqrt{3}\sin\theta) + 12)$$

where $0 \leq \theta \leq 2\pi$.

#5: First of all, note that $0 < 2$, so at the point $(0,2)$ we have $f(x,y) = 1 - |y|$. By definition,

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

So at the point $(0,2)$ we have:

$$\begin{aligned} \frac{\partial f}{\partial x}(0,2) &= \lim_{h \to 0} \frac{f(0+h,2) - f(0,2)}{h} \\ &= \lim_{h \to 0} \frac{(1-|h|) - (1-|0|)}{h} \\ &= \lim_{h \to 0} \frac{-|h|}{h} \end{aligned}$$

As you might remember from single variable calculus, this limit does not exist. If you let $h \to 0^+$ (that is, take the *right hand* limit), it's equal to -1. If $h \to 0^-$ (the *left hand* limit), it's just 1. Since these values do not agree, the limit (and hence the partial derivative) does not exist.

#6: We know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. You can check the following calculations:

$$|\vec{a}| = \sqrt{1^2 + 0^2 + 2^2 + 0^2 + 3^2} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{2^2 + 0^2 + 1^2 + 1^2 + 1^2} = \sqrt{7}$$

$$\vec{a} \cdot \vec{b} = 1 \cdot 2 + 0 \cdot 0 + 2 \cdot 1 + 0 \cdot 1 + 3 \cdot 1 = 7$$

$$\cos\theta = \frac{7}{\sqrt{14}\sqrt{7}} = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4} \text{ or 45 degrees}$$

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