#1. You can do this problem by writing down a few matrices, doing one bit of matrix multiplication, and writing down a formula. I'll explain it in lots of detail here, though, for people who were confused about it.

According to the chain rule,

$$J_h(1,1) = J_g(f(1,1)) \cdot J_f(1,1)$$

We're given J_g; if we evaluate it at the point $f(1,1) = (0,1,-2)$, we get

$$\begin{bmatrix} 1 & 12 & -12 \end{bmatrix}$$

Next we need to find $J_f(1,1)$. f is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$, so its Jacobian should be a 3x2 matrix, namely

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

If you do these derivatives correctly, you'll get

$$\begin{bmatrix} 1 & -1 \\ 3x^2 & 0 \\ 0 & -2 \end{bmatrix}$$

Evaluated at the point $(1,1)$, this becomes

$$\begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Now we can compute down $J_h(1,1)$. It's

$$J_h(1,1) = J_g(f(1,1)) \cdot J_f(1,1) = \begin{bmatrix} 1 & 12 & -12 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 37 & 23 \end{bmatrix}$$

The linear approximation to h at $(1,1)$ is $L(x,y) = h(1,1) + J_h(1,1) \cdot (x-1, y-1)$, or

$$L(x,y) = 5 + \begin{bmatrix} 37 & 23 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = 37x + 23y - 55$$

This problem was similar to #7 on the sample final. (So it stands to reason that something similar could appear on this semester's final.)

#2. The directional derivative is given by $D_{\vec{u}}f(2,0) = \bigtriangledown f(2,0) \cdot \vec{u}$, where $\vec{u}$ is a *unit* vector in the desired direction. In our case,

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

Also, the gradient of $f(x,y)$ is $\bigtriangledown f(x,y) = (\mathrm{e}^y, x\mathrm{e}^y - 1)$, and $\bigtriangledown f(2,0) = (1, 2-1) = (1,1)$. Hence

$$D_{\vec{u}}f(2,0) = \bigtriangledown f(2,0) \cdot \vec{u} = (1,1) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{7}{5}$$

Note that the directional derivative is a *number*, not a vector. If your answer was a vector, you would have lost points here. This problem was similar to example 4.1.2, or any of the exercises 13-23 in section 4.1.

#3. You can immediately say that F is a conservative vector field everywhere on $\mathbb{R}^2$, because it's the gradient of $f(x,y) = x\mathrm{e}^y - y$. (Note that you should have computed this gradient in #2, so the intent was that you get this for free.) Hence, by the Fundamental Theorem of Line Integrals,

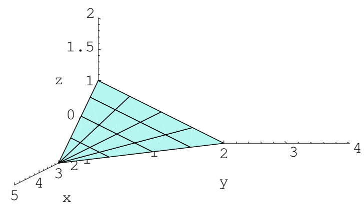
$$\begin{aligned}\int_C F \cdot \vec{x} = \int_C \bigtriangledown f \cdot \vec{x} &= f(\text{ending point of } C) - f(\text{beginning point of } C) \\ &= f(1,1) - f(0,0) = \mathrm{e} - 1\end{aligned}$$

That's it – nothing more was required. Note that *you can't use Green's Theorem to evaluate this integral, because the curve is not closed.* Also, if you try to compute this integral directly (according to the definition), you'll have to integrate e^{t^2}. You might remember from your Calc I days that this can't be done, at least in terms of functions that we know how to write down.

This problem was intended to be easier than your homework problems from section 6.1, because you didn't have to test the vector field for path independence. Unfortunately our hint wasn't clear enough, and many people did more work than they had to here.

The Fundamental Theorem of Line Integrals will return on the final exam, so please ask your lecturer or TA for help if you don't think you understand it fully.

#4. The region of integration is the tetrahedron bounded by the coordinate planes ($x = 0, y = 0, z = 0$) and the plane $2x+3y+6z = 6$. Here's a picture generated by Mathematica:



In the new order, the integral becomes

$$\int_0^2 \int_0^{1-\frac{y}{2}} \int_0^{3-\frac{3y}{2}-3z} f(x, y, z)\, dx\, dz\, dy$$

This type of problem is usually easier to explain in person, by drawing a few sketches, so ask your TA or lecturer if you'd like help understanding where these bounds come from. If you'd like to work through it yourself, I'd suggest drawing a few two dimensional piectures. For example in the original integral, what is the two dimensional region in the xy-plane? And in the new integral, what is the two dimensional region in the yx-plane?

This type of problem may or may not come back on the final exam, so it might be worth brushing up on it if you had trouble. A tetrahedron is a nice, standard sort of region for triple integral questions. This problem was almost the same as exercise 5.2.26, although that exercise had a different plane. It was also similar to example 5.4.4, and these other exercises from your homework: 5.4.7, 5.4.16, 5.4.26, and 5.3.26. Both lecturers also covered triple integrals where the region of integration is a tetrahedron.

#5. Green's Theorem says that, in this situation,

$$\int_C F \cdot d\vec{x} = \int\int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int\int_R \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(-4xy)\, dA = \int\int_R 4x\, dA$$

You can evaluate this integral in either order – $dx\, dy$ or $dy\, dx$, but I'll do the latter here:

$$\begin{aligned}\int\int_R 4x\, dA &= \int_0^3 \int_{x^2}^{3x} 4x\, dy\, dx = \int_0^3 [4xy]_{y=x^2}^{y=3x}\, dx \\ &= \int_0^3 4x(3x) - 4x(x^2)\, dx = \int_0^3 12x^2 - 4x^3\, dx \\ &= \left[4x^3 - x^4\right]_0^3 = 4(27) - 81 = 108 - 81 = 27\end{aligned}$$

This was a fairly typical Green's Theorem problem. The region is similar than that of exercise 6.2.7, for example. If you had trouble with this problem, you should ask your TA or lecturer for help, so that you ace any Green's Theorem question that might show up on the final exam.

#6. Earlier in this exam you evaluated line integrals by using Green's Theorem and the Fundamental Theorem of Line Integrals. The point of this problem was to compute a line integral using the definition. In fact, that's the only way I know how to do this problem; the vector field is not path independent (the Jacobian is not symmetric), and Green's Theorem doesn't apply (because the curve is not closed).

If we parametrize C by $f(t) = (\cos t, \sin t)$, where $0 \le t \le \pi/2$, then:

$$\begin{aligned}
\int_C F \cdot d\vec{x} &= \int_0^{\pi/2} F(f(t)) \cdot f'(t)\, dt = \int_0^{\pi/2} (-\sin t, 0) \cdot (-\sin t, \cos t)\, dt \\
&= \int_0^{\pi/2} \sin^2 t\, dt = \int_0^{\pi/2} \frac{1}{2}(1 - \cos(2t))\, dt = \frac{1}{2}\left[t - \frac{\sin(2t)}{2}\right]_0^{\pi/2} \\
&= \frac{1}{2}\left(\frac{\pi}{2} - 0 - 0 + 0\right) = \frac{\pi}{4}
\end{aligned}$$

The trig identity used in this problem was given to you on the cover page of the exam. A nearly identical example, with the vector field $F = (0, x)$, was done in each lecture, and is posted online in both classes' lecture notes, if you'd like to compare.