Note: most of the problems on this midterm were either homework problems or variations of homework problems. I've noted the origin of each problem here.

#1. (Exercise 6.3.1) The Divergence Theorem says that the flux is equal to the triple integral of the divergence of $\mathbf{F}$ over the interior of the box. In other words, if we call the box S,

$$\iint_M \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_S \operatorname{div}\mathbf{F}\,dV = \int_{-1}^{1}\int_{1}^{4}\int_{-3}^{5} 2y\,dz\,dy\,dx = \int_{-1}^{1}\int_{1}^{4} 16y\,dy\,dx$$
$$= \int_{-1}^{1}\left[8y^2\right]_{y=1}^{y=4}\,dx = \int_{-1}^{1} 120\,dx = 240$$

Your integral might have looked a bit different – you could have set it up in a different order – but your final answer should have been the same.

#2: (Exercise 5.8.6) The most reasonable way to do this problem is probably with spherical coordinates. You could have written down the integral in terms of x, y, and z, and then converted to spherical, or—if you've had enough practice—just started out in spherical coordinates. The most common mistakes on this problem were (a) incorrect bounds for ϕ or θ, and (b) an incorrect "fudge factor," i.e. the change of variables term $\rho^2\sin\phi$. Note that $\sqrt{x^2+y^2+z^2} = \rho$, so

$$\begin{aligned}
\int\int\int_S \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \rho^3\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\rho}{4}\sin\phi\right]_{\rho=0}^{\rho=3} d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\frac{81}{4}\sin\phi\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\frac{81}{4}\left[-\cos\phi\right]_0^{\phi=\pi/2}\,d\theta \\
&= \int_0^{\pi/2}\frac{81}{4}\,d\theta \\
&= \frac{81}{4}\cdot\frac{\pi}{2} = \frac{81\pi}{8}
\end{aligned}$$

#3 (Based on Exercise 5.5.10; also similar to examples and exercises in 5.7) The only thing missing from the integral is the change of variables factor, or "fudge factor." This is the absolute value of the determinant of the Jacobian of the transformation. That's an awfully long phrase, but the work isn't as bad. Since $x = st$ and $y = s/t$,

$$\begin{aligned}\left|\frac{\partial(x,y)}{\partial(s,t)}\right| &= \left|\det\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{pmatrix}\right| = \left|\det\begin{pmatrix} t & s \\ 1/t & -s/t^2\end{pmatrix}\right| \\ &= \left|t\cdot\frac{-s}{t^2} - s\cdot\frac{1}{t}\right| = \left|\frac{-s}{t} - \frac{-s}{t}\right| = \left|\frac{-2s}{t}\right| \\ &= \frac{2s}{t}\end{aligned}$$

There was another, much harder, solution. You could also use the fact that

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{\frac{\partial(s,t)}{\partial(x,y)}}$$

Then you can calculate that

$$\begin{aligned}\left|\frac{\partial(s,t)}{\partial(x,y)}\right| &= \left|\det\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial t}\end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} \\ \frac{1}{2\sqrt{xy}} & -\frac{1}{2}\sqrt{\frac{x}{y^3}}\end{pmatrix}\right| \\ &= \left|-\frac{1}{4|y|} - \frac{1}{4|y|}\right| = \frac{1}{2y} = \frac{t}{2s}.\end{aligned}$$

Therefore, the fudge factor is

$$\frac{1}{\left|\frac{\partial(s,t)}{\partial(x,y)}\right|} = \frac{2s}{t}.$$

#4. (Exercise 5.6.12) We'll compute this integral according to the definition. First note that

$$F(f(s,t)) = (s+t-s, s-t-s, s-t-s-t) = (t, -t, -2t)$$

Also, if you compute the cross product $f_s \times f_t$, you get

$$f_s \times f_t = (1,1,1) \times (-1,1,0) = (-1,-1,2)$$

We're supposed to orient the surface such that the first component of this cross product is *positive*, so we'll actually use $f_t \times f_s = (1,1,-2)$. Now we have all the information we need to set up the surface integral:

$$\begin{aligned}\iint_M \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int_0^1\int_0^1 \mathbf{F}(f(s,t))\cdot(f_t\times f_s)\,ds\,dt = \int_0^1\int_0^1 (t,-t,-2t)\cdot(1,1,-2)\,ds\,dt \\ &= \int_0^1\int_0^1 t - t + 4t\,ds\,dt = \int_0^1\int_0^1 4t\,ds\,dt = \int_0^1 4t\,dt \\ &= \left[2t^2\right]_0^1 = 2.\end{aligned}$$

#5 (Exercise 6.4.1) Using Stokes Theorem, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_M \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where M is any surface which has C as a positively oriented boundary. The given curve C is a circle of radius 1 in the plane $z = 4$, so the simplest choice for M is probably the *disk* of radius 1 in the plane $z = 4$. We need to choose the upward pointing normal vector on this disk if C is going to be the positively oriented boundary of M. (Draw a picture and check this!)

We can parametrize the disk by $f(r,t) = (r\cos t, r\sin t, 4)$, where $0 \le r \le 1$ and $0 \le t \le 2\pi$. If we compute the normal vector we have

$$f_r \times f_t = (0, 0, r)$$

which is the upward pointing normal vector because $r \ge 0$. We also need to compute $\operatorname{curl} \mathbf{F}$, which turns out to be constant: $\operatorname{curl} \mathbf{F} = (2, -5, -2)$. That's all we need to compute the integral.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_M \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \int_0^1 (2, -5, -2) \cdot (0, 0, r)\, dr\, dt \\ &= \int_0^{2\pi} \int_0^1 -2r\, dr\, dt = \int_0^{2\pi} \left[-r^2\right]_{r=0}^{r=1} dr\, dt \\ &= \int_0^{2\pi} -1\, dt = -2\pi\end{aligned}$$

#5: (Exercise 5.5.3, slightly modified) There are many different ways to parametrize this surface. The most common choice on the exam was probably something like this:

$$f(x,t) = (x, \sqrt{5}\cos t, \sqrt{5}\sin t)$$

where $0 \le t \le 2\pi$ and $-2 \le x \le 3$.

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